Monetary Growth Theory

Money, interest, prices, capital, knowledge, and economic structure over time and space

Wei-Bin Zhang

Routledge International Studies in Money and Banking

Monetary Growth Theory

Economic life is changing. A comprehensive study of the rules and functions of money is essential for understanding modern globalizing economies. One needs dynamic theories to comprehend the changing reality. Nevertheless, many economic dynamic models omit monetary issues by explicitly or implicitly assuming that transactions on the economy's real side can be carried out without money. This book answers some challenging questions in monetary growth theory within a compact theoretical framework.

The author succeeds in integrating the theory of money, the theory of value and the theory of growth. The book re-examines many important ideas in modern monetary economics within a single analytical framework. It is concerned not only with traditional one-sector growth models of a homogeneous population with endogenous capital and knowledge, but also with multi-sector models, economies with heterogeneous households, and economies with urban structures, interregional interactions and international trade.

Zhang's book will appeal to those studying monetary economics, neoclassical growth theory, development economics and international economics. It is also useful, more generally, for researchers in social sciences with an interest in the role of money in modern societies.

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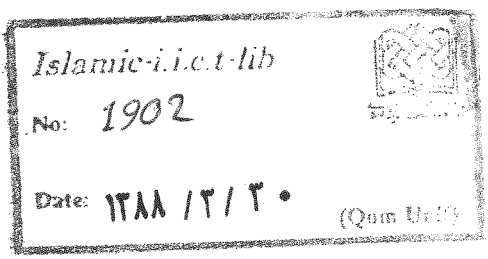

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Preface

This book proposes a monetary growth theory, which integrates economic growth, monetary, and value theory by a general-equilibrium analysis of the commodity and bonds markets over time and space. Two – the Solow and the Ramsey – analytical frameworks in the last five decades, have dominated economic growth theory. The two modeling frameworks have co-existed in "harmony" mainly because one cannot effectively replace the other. The Solow analytical framework is empirically friendly and easy to analyze but it lacks sound behavioral mechanism. The Ramsey analytical framework does seem to be theoretically sound, but it lacks empirical support. A model based on the Ramsey approach tends to become analytically intractable when economic issues related heterogeneous households, or multiple sectors, or urban structure, or interregional interactions are introduced. I have recently proposed an alternative approach to household decision makings to the traditional approaches in dynamic analysis. This approach has been applied to integrate various economic theories into a dynamical whole. It has been systematically applied to (non-monetary) growth theory, urban economics, interregional economics and international trade. It examines not only many traditional economic theories such as the Walrasian equilibrium theory, the Solow growth model, the Keynesian economics, and the Uzawa two sector model, in a consistent and connected manner, but also studies some economic phenomena, such as interregional economic growth with capital accumulation, which are difficult to analyze with the traditional economic models. This book is concentrated on applying this analytical framework to examine economic dynamics with money.

Money, like blood in the human body, affects and is affected by almost all aspects of the economic system. Forms of its existence and functions can exhibit a great variety, depending on, for instance, technology, economic developmental stages, institutional structures, and man's attitude towards the future. Money does not only provide services of the present but also plays the role of a connector between the present and the future. Many of the most intriguing and important questions in dynamic economic analysis involve money. Nevertheless, many economic dynamic models omit monetary issues, by explicitly or implicitly assuming that transactions on the economy's real side can be carried out frictionless without money. In the best developed model of a competitive economy – the Arrow-Debreu framework – there is no role for money. Although money has been introduced into

the neoclassical growth theory initially by Tobin and Sidrauski, many important issues such as growth with heterogeneous households and economic development with urban structure have not been properly examined. This book tries to answer some challenging questions in monetary growth theory.

I would like to thank my Editor Terry Clague, Project Manager Kavitha Ashok, Editorial Assistant Sarah Hastings, and copyeditor Kelly Hallett at Routledge for effective co-operation. I am grateful to Professor Joao Ricardo Faria, whose comments were important for helping to improve the book. I completed this book at the Ritsumeikan Asia Pacific University (APU), Japan. I am grateful to the university's free academic environment. The book is partly based on my research supported by APU Academic Research Subsidy FY2007, which is gratefully acknowledged.

Wei-Bin Zhang
Beppu, 2008

1 Money and growth theory

This book deals with monetary economic growth with an alternative approach to consumer behavior proposed by Zhang (2005a, 2006a). In almost all the parts of Zhang's two books, the functioning of money is otiose because the functions of money and finance are performed by the markets themselves. While goods are exchanged and transactions must be taking place, there is no medium of exchange that is used to facilitate these transactions. The theory does not involve any asset, like money, that has a zero nominal rate of return and is therefore dominated in rate of return by other interest-bearing assets. This book constructs a monetary economic growth. We first briefly introduce the literature of money in economic growth theory. The contemporary literature of monetary growth theory will be examined in details in the main context.

1.1 Money in economic growth theory

Money is an important part of economic systems. In economic theory, money is often defined in terms of the functions that it performs.[1] The traditionally accepted roles of money are a unit of account, a means of exchange, standard of deferred payments, and a store of value. Of these functions, the medium of payments is the essential function of money. A modern economy usually has many assets which can perform this role.[2]

It is generally accepted that the modern monetary theory began from David Hume's essays of 1752, *Of Money and Interest*.[3] Hume held that the effect of changes in money should depend on the way in which the change is affected. For instance, he stressed the units-change aspect of changes in the money stock and the irrelevance of such changes to the behavior of rational people. He also observed that it would encourage industrial development by properly increasing the money supply. He argued that when the money is increased ...

> at first, no alternation is perceived; by degrees the price rises, first of one commodity, then of another; until the whole at last reaches a just proportion with the new quantity of specie which is in the kingdom. ... The good policy of

> the magistrate consists only in keeping it, if possible, still encreasing; because by that means, he keeps alive a spirit of industry in the nation.[4]

As the money supply is increased, prices rise until they reach a new level proportional to the new money supply. During the period in which prices rise output will be increased. As prices reach the new equilibrium level, output is back to normal. Hume held that the national spirit of industry can be maintained by timely increasing money.

Keynes intensified debates over money after he published *General Theory* in 1936. He held that a fully competitive economy could get trapped into a disequilibrium situation (with unemployment). Pigou (1943) did not agree with Keynes' arguments, holding that Keynes overlooked an important class of regulating mechanisms, namely, the real balance, or wealth effects, which are interacting with nominal prices, wages and interest rates.[5] As pointed out by Grandmont (1983: 1)

> A broad agreement was reached in the 1950s, known as "neoclassical synthesis": If such wealth effects were properly integrated in the analysis, the full price flexibility ... was bound to remove all excess demands and supplies, both in the short run and in the long run. Keynes was theoretically mistaken, and the unemployment he talked about was entirely due to his assumption that nominal wages were rigid down.

Friedman (1964) describes a trend in the post war period in monetary theory and policy as follows:

> The post-war period has seen a dramatic change in the views of academic students of economics about monetary theory and of government officials about monetary policy. At the end of the war most professional economists and most governmental officials concerned with economic policy took it for granted that money did not matter, that it was a subject of minor importance. Since then, there has been something of a counter-revolution in both theory and policy.

In fact, economic theory found some difficulties in incorporating money, especially under the form of fiat money,[6] into general equilibrium models. To analyze monetary issues, it is necessary to specify a role for money so that the agents will wish to hold positive quantities of money. Money is a peculiar 'commodity' in that its usefulness depends on its price. As Starr (1974: 45) points out, "It would not upset the theory of value if water or diamond had a price of zero, but monetary theory depends on money having a positive value in exchange." A positive demand for money is necessary if money is to have positive value. In other words, the money price of goods should be bounded. Let P denote the price of goods in terms of money. Then 1 unit of money will purchase $1/P$ units of goods. If money has positive value, i.e. $1/P > 0$, then P is bounded. The problem of why money has

positive value is called the *Hahn problem.*[7] Hahn (1965) studies a one-period competitive general equilibrium model which was based on a model by Patinkin (1956), showing that the economic system always has a non-monetary solution, where fiat money has zero price. This implies that money cannot perform any of its main functions.

In order to introduce a role for money, the economic environment has to be specified. In the classical literature, money is primarily regarded as a medium of exchange and unit of account. Money was held for the commodities and services that it could buy. As income is closely related to the purchases of goods and services, the demand for money should also be closely related to the income. Keynes assumed the transactions demand for money as a function of nominal income

$$M^d = \kappa PY,$$

where P is the aggregate price level, Y is the level of real output and κ is a positive parameter. This relation means that to spend one needs cash. This relation can also be interpreted as that cash holdings is for smoothing out the difference between frequent income receipts and expenditure payments.

Different authors emphasize different aspects of money with different modeling frameworks. For instance, with regard to effects of a shift to a fully anticipated, more inflationary policy on capital accumulation, economists answer quite differently. For instance, in his well-known paper on long-run effects of inflationary policies, Tobin (1965) showed that an increase in the level of the inflation rate will increase the capital stock of an economy. Sidrauski (1967a) constructed an economic model in which no real variable will be affected in the economy's inflation rate. Stockman (1981) presented another influential economic model which forecasts negative long-run impact of inflation on the real variables. Since the classical papers by Tobin and Sidrauski in the 1960s, the analysis of the effects of anticipated inflation on capital accumulation has been a central issue in monetary growth theory. In the contemporary literature of monetary economics, three approaches to incorporating money into general equilibrium models are mostly employed:[8] (1) assume that money yields direct utility by introducing money directly into the utility functions of the agents (Sidrauski, 1967a); (2) consider transactions costs of some form that yields a demand for money, either by making asset exchanges costly (Baumol, 1952; Tobin, 1956), requiring that money be employed by some types of transactions (Clower, 1967), or assuming that direct barter of commodities is costly (Kiyotaki and Wright, 1989); or (3) use money like any other asset for transferring resources intertemporally (Samuelson, 1958). The following state of macroeconomics described by Gale (1983: 1) more than two decades is still valid for the current state of monetary economics:

> Patinkin's great work in the fifties, for example, was a serious attempt at placing macroeconomics in a general-equilibrium framework. Likewise,

> the work by countless authors on the theory of the consumption and investment functions was a serious attempt at grounding elements of the IS-LM framework in microeconomic principles. But these attempts, valuable though they were, were piecemeal and did not go far enough.

As far as monetary growth theory is concerned, monetary growth theory is piecemeal and fails to study effectively various issues of economic growth in a consistent manner. Orphanides and Solow (1990: 258) also point out:

> The fundamental difficulty is that we do not yet have any clearly preferred way to introduce money into models of the real economy, especially those that feature durable productive assets as well. Models of a monetary economy without real capital cannot be taken seriously as vehicles for the study of money-and-growth. Their implications may not serve the introduction of productive assets; in that case they serve only to mislead. That may well be the case with extant OLG models.

As shown later on, almost all monetary growth models are developed either in the Ramsey framework or its discrete version (OLG models). Nevertheless, the Ramsey approach to household behavior has neither solid theoretical basis nor empirical supports, and the history of economic analysis in the last 50 years has demonstrated that the Ramsey approach does not provide an effective tool for analyzing many important issues of economic growth with money. It is challenging to propose an alternative approach to traditional theories so that the basic issues in economic growth theory can be treated in a comprehensive and consistent manner.

Rather than being broadly concerned with monetary economics, this book is focused in monetary economic growth, emphasizing price-level determination, inflation, the role of monetary policy, and economic growth.[9] The main contribution of this book is to re-examine many important ideas in modern monetary economics within a single analytical framework. We examine relationships between real and nominal economic variables.[10] We are concerned not only with traditional one-sector growth models with a homogeneous population, but also with multi-sector growth models and dynamic economies with heterogeneous households.

The study attempts to integrate the theory of money, the theory of value, and the theory of growth. We examine questions related to functions that money performs in economic systems. The analytical approach is influenced by the Walrasian general equilibrium theory, the neoclassical growth theory, and the contemporary literature of monetary growth theory. We provide a dynamic framework to introduce money into growth theory. In general, money is considered to be any good or token that fulfills the money functions, a medium of exchange, store of value, and unit of account. In this study, money means currency. Currency or fiat money is a kind of public liability, denominated in arbitrary units of account. Currency does not bear interest. The adjective "fiat"

is related to government permission or decree and implies a liability that is not convertible into or backed by commodities. The theoretical models in this book abstract from the banking system and any devices, such as checks and credits, which are actually widely used to facilitate transactions. By focusing the narrow interpretation of money, we can make our analysis simple and their implications transparent. Most of our discussions assume that each national government is a monopoly issuer of the currency used in domestic transactions. The assumption of a government monopoly over currency issue is sometimes not realistic. Contemporary economic reality of money is characterized of a swamp of institutional details. Nevertheless, in order to comprehend the performance of real economic systems it is important to analyze clear, explicit and consistent models at first.

1.2 The one-sector growth (OSG) model[11]

The main deviation of this book from traditional approaches in dynamic economics is how to model households' decision making. As this approach is essential for understanding and appreciating the book, we will introduce the approach in detail. For illustration, we are concerned with economic growth without money. We start investigating economic growth with an economy of one production sector and one type of households. We introduce a number of ideas basic to the later discussion. In the rest of this book, we use the OSG framework to stand for the one sector growth model developed in this section and its variation extensions. With regard to production, almost all the aspects of the OSG model are similar to the standard Solow and Ramsey one-sector growth models. It is assumed that there is only one (durable) good in the economy under consideration. Households own assets of the economy and distribute their incomes to consume and save. Production sectors or firms use inputs such as labor with varied levels of human capital, different kinds of capital, knowledge and natural resources to produce material goods or services. Exchanges take place in perfectly competitive markets. Production sectors sell their product to households or to other sectors and households sell their labor and assets to production sectors. Factor markets work well. Saving is undertaken only by households, which implies that all earnings of firms are distributed in the form of payments to factors of production, labor, managerial skill and capital ownership.

First, we describe behavior of the production sector.[12] Time is represented continuously by a numerical variable which takes on all values from zero onwards ($t \geq 0$). Let $K(t)$ denote the capital existing at each time t and $N(t)$ the flow of labor services used at time t for production. Capital is malleable in the sense that one need distinguish neither its previous use nor the factor productions of its previous use. We use the conventional production function to describe a relationship between inputs and output. The function $F(t)$ defines the flow of production at time t. The production process is described by neoclassical production function:[13] $F(t) = F(K(t), N(t))$. In the rest of this book, we omit time subscripts whenever no ambiguity results.

We assume that the production function exhibits constant returns to scale.[14] It is direct to check that a linear homogenous production $F(K,N)$ has the following properties:

(i) The production function can be written in terms of per capita output as a function of per capita capital

$$\frac{F}{N} = F(k,1) \equiv f(k), \quad k \equiv \frac{K}{N}. \tag{1.2.1}$$

Output per worker depends only on the amount of capital employed by one worker. Equation (1.2.1) is called the intensive form of the aggregate production function. It is also referred to as the per-worker production function.

(ii) The slope of $f(k)$ represents the marginal product of capital, i.e.

$$F_K = \frac{\partial F}{\partial K} = \frac{\partial (F/N)}{\partial (K/N)} = f'(k) > 0.$$

(iii) The marginal product of labor can be obtained by $f(k)$ as follows

$$F_N = \frac{\partial F}{\partial N} = \frac{\partial (NF/NK)}{\partial (N/K)} = \frac{\partial (f(k)/k)}{\partial (1/k)} = f(k) - kf'(k) > 0.$$

(iv) The Euler Theorem holds: $KF_K + NF_N = F$.

We depict intensive form $f(k)$ of the aggregate production function in Figure. 1.2.1. As we move out to the right along the production function, the output per worker increases as the capital/labor ratio k rises. The shape of $f(k)$ in the figure reflects the assumption that there are diminishing returns to increases in k. The increment to output per worker declines as capital per worker rises. The slope of the production function becomes flatter from left to right. This means that although more capital always leads to more output, it does so at a decreasing rate.

We assume (identically numerous) one production sector. Its goal of economic production is to maximize its current profit

$$\pi(t) = p(t)F(t) - (r(t) + \delta_k)p(t)K(t) - w(t)p(t)N(t),$$

where $p(t)$ is the price of product, $r(t)$ is the real rate of interest, $w(t)$ is the real wage rate, and δ_k is the fixed depreciation rate of capital. We assume that the output good serves as a medium of exchange and is taken as numeraire. We thus set $p(t) = 1$ and measure both wages and rental flows in units of the output good. The rate of interest and wage rate are determined by markets. Hence, for any individual firm r and w are given at each point of time. The production sector chooses the two

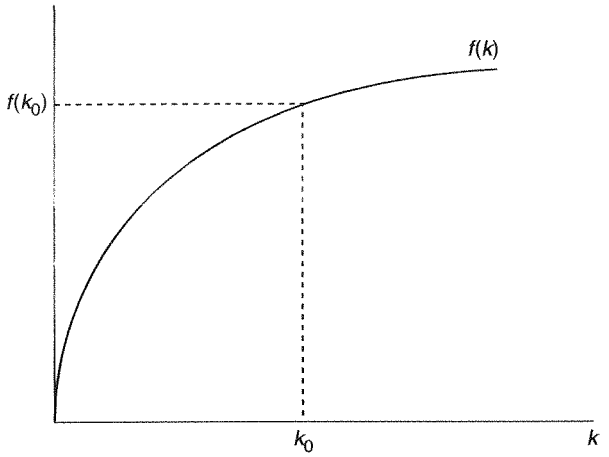


Figure 1.2.1 Intensive form of the aggregate production function.

variables K and N to maximize its profit. Maximizing π with regards to K and N as decision variables yields

$$r+\delta_k = F_K = f'(k), \quad w = F_N = f(k) - kf'(k), \tag{1.2.2}$$

in which $k(t) \equiv K(t)/N(t)$. We assume that factor markets work quickly enough so that our system always displays competitive equilibrium in factor markets. Since we assumed that the production function is homogenous of degree one, we have $KF_K + NF_N = F$, or

$$rK + wN + \delta_k K = F. \tag{1.2.3}$$

This result means that the total revenue is used up to pay all factors of the production.

We now describe behavior of consumers. Consumers obtain income

$$Y = rK + wN, \tag{1.2.4}$$

from the interest payment rK and the wage payment wN. We call Y the current income in the sense that it comes from consumers' daily toils (payment for human capital) and consumers' current earnings from ownership of wealth. The current income is equal to the total output as we neglect any taxes at this initial stage. The sum of income that consumers are using for consuming, saving, or transferring are not necessarily equal to the temporary income because consumers can sell wealth to pay, for instance, the current consumption if the temporary income is not sufficient for buying food and touring the country. Retired people may live not only on the interest payment but also have to spend some of their wealth. The total

value of wealth that consumers can sell to purchase goods and to save is equal to $p(t)K(t)$ (with $p(t) = 1$). Here, we assume that selling and buying wealth can be conducted instantaneously without any transaction cost. The disposable income is equal to

$$\hat{Y}(t) = Y(t) + K(t). \tag{1.2.5}$$

It should be noted that in equation (1.2.5), like $Y(t)$, the value of wealth, $K(t)$, is a flow variable. The disposable income is used for saving and consumption.

At each point of time, consumers would distribute the disposable income between saving, $S(t)$, and consumption of goods, $C(t)$. The budget constraint is given by

$$C(t) + S(t) = \hat{Y}(t). \tag{1.2.6}$$

In our model, at each point of time, consumers have two variables to decide. A consumer decides how much to consume and to save. Equation (1.2.6) means that consumption and savings exhaust the consumers' disposable personal income. The slope of the budget line is equal to -1, i.e. $dS/dC = -1$.

We assume that utility level, $U(t)$, that the consumers obtain is dependent on the consumption level of commodity, $C(t)$, and the saving, $S(t)$

$$U(t) = U(C(t), S(t)). \tag{1.2.7}$$

A typical consumer is to choose his most preferred bundle $(c(t), s(t))$ of consumption and saving under his budget constraint. Here, $c \equiv C/N$ and $s \equiv S/N$. The utility maximizing problem at any time is defined by

$$\begin{aligned} &\underset{c,s\geq 0}{Max}\, U(c(t), s(t)) \\ &\text{s.t. } c(t) + s(t) \leq \hat{y}(t), \end{aligned} \tag{1.2.8}$$

in which $\hat{y}(t) \equiv Y(t)/N(t) + k(t)$. The following theorem holds.

Proposition 1.2.1
Let $U(c, s) : R^2_+ \to R^1$ be a C^1 function that satisfies the monotonicity assumption, which says that $\partial U/\partial c > 0$ and $\partial U/\partial s > 0$ for each (c, s) satisfying the constraint set in problem (1.2.8). Suppose that (c^*, s^*) maximizes U on the constraint set. Then, there is a scalar $\overline{\lambda}^* > 0$ such that

$$\frac{\partial U}{\partial c}(c^*, s^*) \leq \overline{\lambda}^*, \quad \frac{\partial U}{\partial s}(c^*, s^*) \leq \overline{\lambda}^*.$$

We have $\partial U/\partial c = \overline{\lambda}^*$ if $c^* \neq 0$ and $\partial U/\partial s = \overline{\lambda}^*$ if $s^* \neq 0$. If both $c^* > 0$ and $s^* > 0$, then

$$\frac{\partial U}{\partial c}(c^*, s^*) = \overline{\lambda}^*, \quad \frac{\partial U}{\partial s}(c^*, s^*) = \overline{\lambda}^*.$$

Conversely, suppose that U is a C^1 function, which satisfies the monotonicity assumption and that $(c^*, s^*) > 0$ and the first order conditions. If U is C^2 and if

$$|\overline{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & U_{cc} & U_{cs} \\ 1 & U_{sc} & U_{ss} \end{vmatrix} = 2U_{cs} - U_{cc} - U_{ss} > 0,$$

then (c^*, s^*) is a strict local solution to the utility maximization problem. If U is quasiconcave and $\nabla U(c, s)$ for all $(c, s) \neq (c^*, s^*)$, then (c^*, s^*) is a global solution to the problem.

The proof of this proposition and other general properties of the problem can be found in standard textbooks of microeconomics or mathematical economics.[15] We require that U is a C^2 function, and satisfies $U_c > 0$, $U_s > 0$ for any $(c, s) > 0$. It can be shown that that $0 < ds/d\hat{y} < 1$ and $0 < dc/d\hat{y} < 1$ in the case of $U_{sc} \geq 0$ under the second-order condition of maximization. We denote an optimal solution as function of the disposable income

$$(c(t), s(t)) = (c(\hat{y}(t)), s(\hat{y}(t))).$$

The vector $(c(\hat{y}(t)), s(\hat{y}(t)))$ is known as the Walrasian (or ordinary or market) demand function, when it is single-valued for all positive disposable income.

It appears reasonable to consider population as independent of economic conditions, as a first approximation. We assume that the population dynamics is exogenously determined in the following way

$$\dot{N}(t) = nN(t),$$

where n is a constant growth rate of N.

The change in the households' wealth is equal to the net savings minus the wealth sold at time t, i.e.

$$\dot{K}(t) = s(\hat{y}(t))N(t) - K(t). \tag{1.2.9}$$

Inserting

$$\dot{k}(t) = \frac{\dot{K}(t)}{N(t)} - nk(t),$$

in equation (1.2.9) yields

$$\dot{k}(t) = s(\hat{y}(t)) - (1+n)k(t),$$

where

$$\hat{y}(k(t)) = \frac{\hat{Y}(t)}{N(t)} = f(k(t)) + k(t).$$

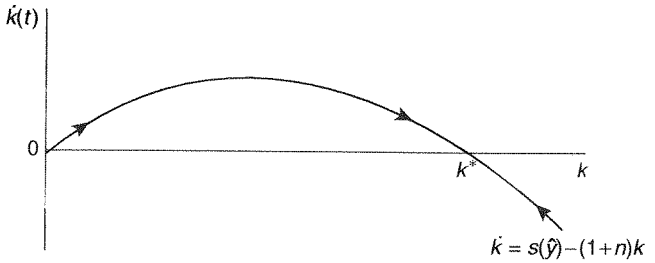


Figure 1.2.2 Evolution of capital–labor ratio in the OSG model.

In a stationary state, we have

$$s(\hat{y}(k)) = (1+n)k.$$

It can be shown that this equation has a unique solution. The following theorem is proved in Zhang (2006a: Chapter 2).

Theorem 1.2.1
Given a neoclassical production function and a utility function that is a C^2 function, and satisfies $U_c > 0, U_s > 0$ for any $(c(t), s(t)) > 0$. Let the bordered Hessian be positive for any nonnegative $(c(t), s(t))$. Then the capital–labor ratio converges monotonically to a unique positive steady state. The unique stationary state is stable.

The stability guaranteed above is local. We now show that if $s(\hat{y})$ is concave in $\hat{y}$, then the system is globally stable. Because of $d^2c/dy^2 = -d^2s/dy^2$ by equation

$$1 = \frac{dc}{d\hat{y}} + \frac{ds}{d\hat{y}},$$

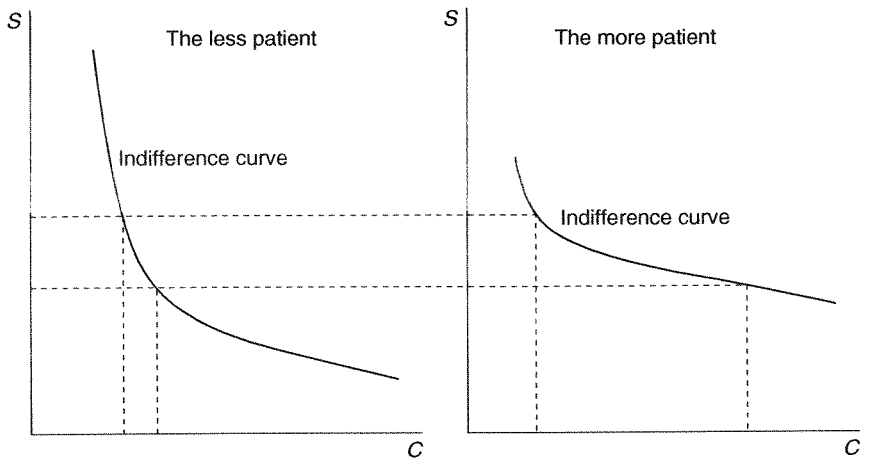
concavity of s implies convexity of c. From the first-order conditions, it is straightforward to give that conditions under which s is concave, we omit the expression because we lack a clear economic interpretation. The asymptotical stability can be confirmed by directly applying Lyapunov's theorem.[16]

We illustrate dynamics of capital–labor ratio in Figure 1.2.2. The dynamic system has a unique stable equilibrium.[17]

1.3 The OSG model with the Cobb-Douglas functions[18]

This section solves the OSG model when the production and utility functions are taken on the Cobb-Douglas functions[19]

$$F(t) = AK^{\alpha}(t)N^{\beta}(t), \quad \alpha, \beta > 0, \quad \alpha + \beta = 1,$$
$$U(t) = C^{\xi}(t)S^{\lambda}(t), \quad \xi + \lambda = 1, \quad \xi, \lambda > 0, \tag{1.3.1}$$



a) $\xi = 0.5, \lambda = 0.5$ b) $\xi = 0.2, \lambda = 0.8$

Figure 1.3.1 Indifference curves and the propensity to hold wealth.

where A is a number measuring overall productivity, and α, β, ξ, and λ are parameters. The marginal conditions for the production sector are

$$r = \alpha A k^{-\beta} - \delta_k, \quad w = \beta A k^{\alpha}, \quad f = A k^{\alpha}. \tag{1.3.2}$$

The preference represented by $U = C^{\xi} S^{\lambda}$ is described in Figure 1.3.1. In Figure 1.3.1a, we illustrate the indifference curve for $\xi = 0.5$ and $\lambda = 0.5$. In Figure 1.3.1b, we illustrate the indifference curves for $\xi = 0.2$ and $\lambda = 0.8$. Different values of the parameters ξ and λ lead to different shapes of the indifference curves. The greater the propensity to own wealth is, the more patient the consumer becomes. The consumer described in Figure 1.3.1a is less patient than the consumer in Figure 1.3.1b.

It is straightforward to solve the optimal choice of the consumers as follows

$$C^*(t) = \xi \hat{Y}(t), \; S^*(t) = \lambda \hat{Y}(t). \tag{1.3.3}$$

The optimal choice is illustrated in Figure 1.3.2.

In Figure 1.3.1, we illustrate how values of the propensity to save λ determine shapes of indifference curve. Figure 1.3.3 illustrates how changes in the value of λ affect optimal solutions. It can be seen that as the consumer's propensity to save becomes higher, out of the same disposable personal income the consumer saves more.

By Figure 1.3.2, we conclude that as $\hat{Y}(t)$ increases, both $C(t)$ and $S(t)$ are increased. An increase in $\hat{Y}(t)$ may be caused either by increases in the interest rate, or the wage rate, or the wealth. If $r(t), w(t)$, or $K(t)$ is increased, both $C(t)$

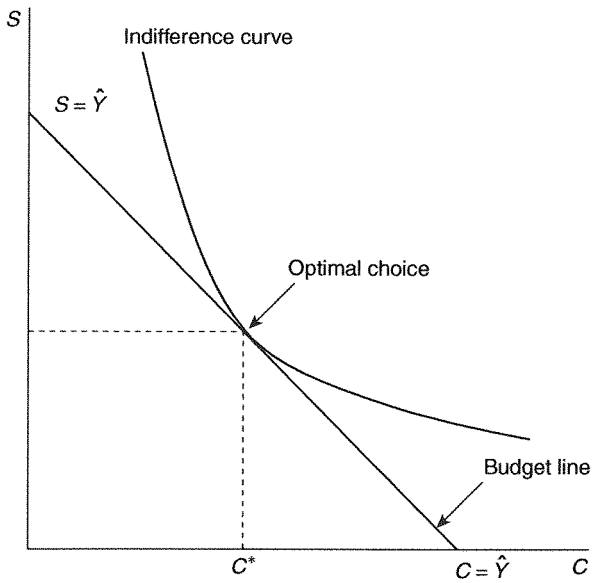


Figure 1.3.2 Optimal choice at time τ.

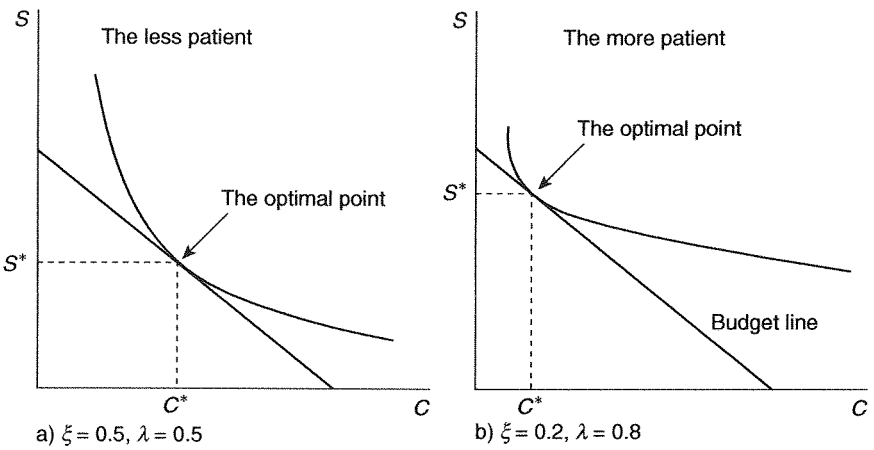


Figure 1.3.3 Optimal savings with different propensities to own wealth.

and $S(t)$ are increased. For instance, as the interest rate increases, the budget line is shifted right-upwards in parallel to the original line, as shown in Figure 1.3.4.

By equations (1.3.3), we have

$$c = \xi(Ak^{\alpha} + k), \quad s = \lambda(Ak^{\alpha} + k).$$

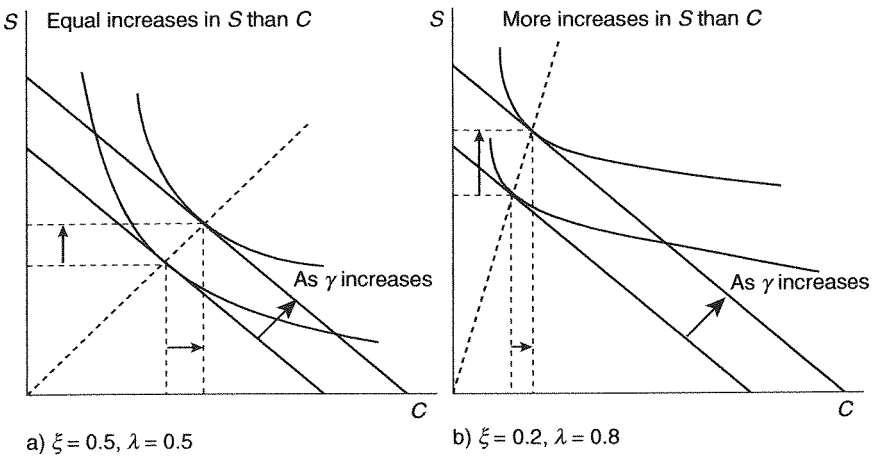


Figure 1.3.4 The impact of changes in the interest rate.

The motion of per-capita capital is given by

$$\dot{k} = \lambda A k^{\alpha} - (\xi + n)k. \tag{1.3.4}$$

Equation (1.3.4) is a Bernoulli equation in the variable $k(t)$. Inserting $z(t) = k^{\beta}(t)$ in equation (1.3.4) yields

$$\dot{z} + (1-\alpha)(\xi + n)z = (1-\alpha)\lambda A,$$

which is a standard first-order linear differential equation. The solution is then given by

$$z(t) = \left(z(0) - \frac{\lambda A}{\xi + n}\right) e^{-\beta(\xi+n)} + \frac{\lambda A}{\xi + n}.$$

Substituting $z(t) = k^{\beta}(t)$ back to the solution, we obtain

$$k^{\beta}(t) = \left(k^{\beta}(0) - \frac{\lambda A}{\xi + n}\right) e^{-\beta(\xi+n)t} + \frac{\lambda A}{\xi + n},$$

where $k(0)$ is the initial value of the capital–labor ratio, $k(t)$. This solution is what determines the time path of capital intensity, $k(t)$. Once we determine the value of $k(t)$, we can decide the values of all the other variables at any point of time.

As $t \to +\infty$, the exponential expression will approach zero. Consequently, as $t \to +\infty$ the dynamic system will achieve its unique steady state

$$k^* = \left(\frac{\lambda A}{\xi + n}\right)^{1/\beta}. \tag{1.3.5}$$

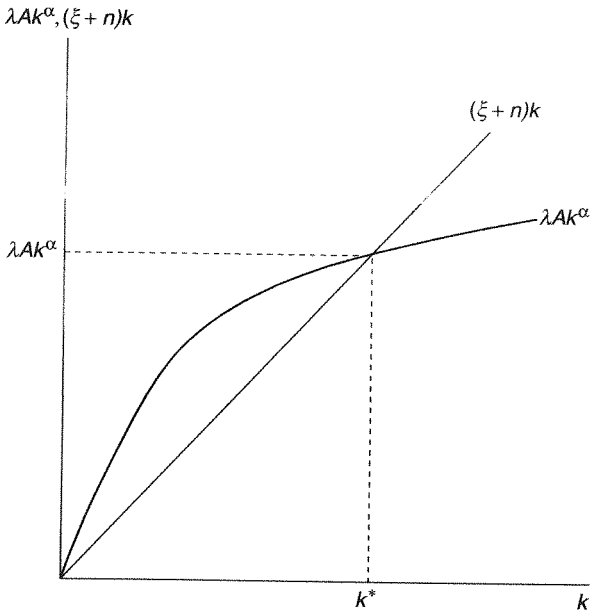


Figure 1.3.5 Evolution of capital–labor ratio in the OSG model.

The capital–labor ratio will approach a constant as its equilibrium value. This steady state varies directly with the propensity to save, λ, the technology, A, and inversely with the propensity to consume, ξ, and the population growth rate, n.

The dynamics is depicted in Figure 1.3.5. Since equation (1.3.4) contains two terms, λAk^{α} and $(\xi + n)k$, we plot them as two separate curves. The $(\xi + n)k$ term, a linear function of $k(t)$, will show up in the figure as a straight line, with a zero vertical intercept and a slope equal to $(\xi + n)$. The term, λAk^{α}, plots as a curve that increases at a decreasing rate. Based upon these two curves, we can measure the value of the change rate $\dot{k}$ for each value of $k(t)$ by the vertical distance between the two curves. The two curves intersect when the capital–labor ratio arrives at steady state. As shown by the figure, the capital–labor ratio will always reach a positive steady state. For instance, if $k(t)$ starts at some value less than k^*, λAk^{α} is greater than $(\xi + n)k$. When this extra amount $\lambda Ak^{\alpha} - (\xi + n)k$ is converted into capital, the capital–labor ratio will rise. As indicated by the arrows, $k(t)$ increases towards k^*. We can similarly explain falling of $k(t)$ when it starts at some value greater than k^*.

The right-hand side of equation (1.3.4) first increases and then decreases, and eventually becomes and remains negative. These imply that there exists a unique $k^* > 0$ such that $k'(t) > 0$ when $k'(t) < k^*$, $k'(t) = 0$ when $k'(t) = k^*$, and $k'(t) < 0$ when $k'(t) > k^*$. Thus k^* is a globally asymptotically stable equilibrium.

To see motion of the growth rate, dividing equation (1.3.4) by $k(t)$, we obtain growth rate $g_k(t)$ of per capita capital as

$$g_k(t) \equiv \frac{\dot{k}(t)}{k(t)} = \lambda A k^{-\beta}(k) - (\xi + n).$$

Here, $g_k(t)$ stands for growth rate of per capita capital at time t. The above equation says that the growth rate of per capita capital equals the difference between two terms, $\lambda A k^{-\beta}$ and $\xi + n$, which we plot against k in Figure 1.3.6. The first curve is a downward-sloping curve and the second term is a horizontal line. The vertical distance between the curve and the line equals the growth rate of per capita capital. As shown before, there is a unique equilibrium. The Figure shows that to the left of the steady state, the curve lives above the line. Hence, the growth rate is positive and k increases over time. As k rises, the growth rate declines. Finally, k achieves at k^* as the growth rate becomes zero. An analogous argument demonstrates that if the system starts from the right of the steady state, the growth rate is negative. As k declines, the growth rate rises and finally becomes zero.

When the economy has reached the stationary capital intensity, capital per capita will remain the same as time passes, but the stock of capital, $K(t)$, remains growing infinitely at the same predetermined rate as the labor force, n. The sustainable growth rate of the model is exogenously given by n. This can be confirmed by

$$K(t) = k^* N_0 e^{nt}, \quad F(t) = f(k^*) N_0 e^{nt}, \quad C(t) = c(k^*) N_0 e^{nt}.$$

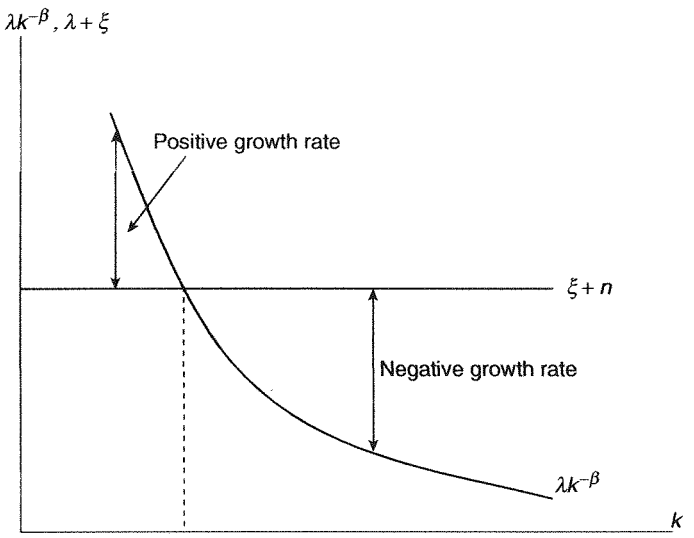


Figure 1.3.6 Dynamics of growth rate in the OSG model.

1.4 Theoretical foundations of the utility function

We now provide some insights into utility functions accepted in the OSG model. We show that it is possible to specify manifested forms of consumers' preference structures by the utility function proposed by this study.[20]

We assume that at any point of time the consumer has preferences over alternative bundles of commodities, which can be divided into goods, services, and time distribution of the consumer. The behavioral rule consists of maximization of these preferences under budgets restrictions of finance, or time, or human capital, or energy.

A commodity is characterized by its location, date at which it is available and its price. At each point of time, the consumer is faced with a commodity bundle consisting of (finite) real numbers

$$\{x_j(t)\}, \quad j = 1, 2, \cdots, m,$$

indicating the quality of each commodity. The commodity space consists of commodity bundles. Here, we omit issues related to spatial location. Let us denote the price of commodity j by $p_j(t)$. For simplicity, we omit time index of x and p except in some circumstances. Both commodity vector x, and price vector p, can be represented by points in Euclidean space R^m, i.e. $x \in R^m$ and $p \in R^m$. The value of the commodity bundle at any point of time is given by $p(t)x(t) = \sum_j p_j(t)x_j(t)$. Obviously, some bundles of the commodity space may be excluded as consumption possibilities. The consumption set, denoted by X, consists of all consumption bundles, which are possible. Let us assume that the consumer's choices are restricted by the fact that the value of his consumption should not exceed his income $w(t)$, at each point of time. The budget set

$$\beta(p, x, t) \equiv \{x \in X \,|\, px \leq w\},$$

is the set of possible consumption bundles whose value does not exceed the income.

The consumer has tastes and desires. These are important in analyzing why the consumer chooses a bundle from the consumption set. Mathematically, we represent the preference structure by the consumer's preference relation, $\succeq_t$, at each point of time which is a binary relation on X. For any two bundles, $x(t)$ and $y(t)$, $x \in X$ and $y \in X$, $x \succeq y$ means that x is at least as good as y at time t. Before discussing the relation between the preference relation and utility functions, we introduce the following axioms.

Axiom 1 (reflexibility)
For all $x \in X$, $x \succeq x$, i.e. any bundle is as good as itself.

Axiom 2 (transitivity)
For any three bundles, x, y, z in X such that $x \succeq y$ and $y \succeq z$ we have $x \succeq z$.

Axiom 3 (completeness)
For any two bundles x and y in X, $x \succeq y$ or $y \succeq z$.

Axiom 4 (continuity)
For every $x \in X$ the upper contour set $\{y \in X \,|\, y \succeq x\}$ and the lower contour set $\{y \in X | x \succeq y\}$ are closed relative to X.

A preference relation $\succeq$ which satisfies the first three axioms is a complete preordering on X and is called a preference order. A bundle x is said to be strictly preferred to a bundle y, i.e. $x \succ y$ iff $x \succeq y$ and not $y \succeq x$. A bundle x is said to be indifferent to a bundle y, i.e. $x \sim y$ iff $x \succeq y$ and $y \succeq x$. The indifference relation defines an equivalent relation on X, i.e. $\sim$ is reflexible, symmetric, and transitive. We always assume that X includes at least two bundles x' and x'' such that $x' \succ y''$. In order to solve the problem of the representability of a preference relation by a numerical function, we introduce the concept of utility function.

Definition 1.4.1
Let X denote a set and $\succeq_t$ a binary relation on X at time t. Then a function u from X into real R is a representation of $\succeq$, i.e. a utility function for the preference relation $\succeq$, if, for any two points x and y, $u_t(x) \geq u_t(y)$ iff $x \succeq ty$ at point of time t.

It seems that Pareto was the first to recognize that arbitrary increasing transformation of a given function would result in identical maximization of a consumer. From the above definition we see that for any utility function u_t and any increasing transformation $f : R \to R$ the function $v_t = f \circ u_t$ is also a utility function for the same preference relation $\succeq$. The following theorem is referred to Debreu or Rader.[21]

Theorem 1.4.1
Let X denote a topological space with a countable base of open sets and $\succeq$ a continuous preference order defined on X, i.e. a preference relation that satisfies Axioms 1–4. Then there exists a continuous function u.

The above theorem shows that under certain conditions the concepts of utility and of the underlying preferences may be used interchangeably to determine demand at any point of time. This theorem is referred to any point of time. It does not involve how to calculate the future. When applying to our cases, we assume that a consumer is to maximize his utility by choosing consumption, $C(t)$, and saving, $S(t)$, with the disposable personal income, $\hat{Y}(t)$, at each point of time t.

We now examine relations of our alternative approach to consumer behavior with some traditional approaches in describing behavior of consumers in economic dynamic analysis.

1.5 The Keynesian consumption function and the OSG approach

The Keynesian theory of consumption is that current real disposable income is the most important determinant of consumption in the short run. Keynes developed

the idea of the consumption function, which considers consumption as consisting of two main parts. The first part is autonomous consumption, which refers to consumption spending done as part of long-term plans for future[22] and as a result of habits and contractual commitments. The other part is induced consumption, which refers to increases in consumer spending as disposable income rises. In the General Theory, Keynes postulated that current consumption expenditure has a stable relation to current income and that a greater proportion of income is saved as real income rises. Keynes in 1936 argued:[23] "... men are disposed, as a rule and on the average, to increase their consumption as their income increases, but not by as much as the increase in their income." This statement by Keynes stimulated many empirical works to derive the consumption function. Many attempts were made to correlate aggregate consumption expenditures over time with aggregate disposable income and other variables. The traditional Keynesian consumption function posits that consumption is determined by current disposable income, i.e.

$$C(t) = a + bY(t), \quad a > 0,\ 0 < b < 1,$$

where a and b are constant, $C(t)$ is real consumption at time t, and $Y(t)$ is real disposable income (which is the same as the current income in our model), which equals GNP minus taxes. It can be seen that if we swap the real disposable income in the Keynesian model with the disposable personal income in our model in Section 1.3, we see that the Keynesian consumption function with $a = 0$ is identical to the consumption derived from our rational choice assumption. It is called autonomous consumption. If a consumer's income feel to zero some of his existing spending could be sustained by using savings. This is known as dissaving. Since Y in the Keynesian consumption function is the current consumption in our model, we cannot see further relations between the two approaches without further exposition.

The parameter, b, is the marginal propensity to consume, which measures the increase in consumption in association with per unit increase in disposable income. The intercept, a, measures consumption at a zero level of disposable income. The Keynesian consumption function is illustrated in Figure 1.5.1.

Because of the intercept, the Keynesian consumption is not a proportional relationship between consumption and income. The ratio of consumption to income is termed the average propensity to consume (APC), i.e.

$$APC = \frac{C}{Y} = b + \frac{a}{Y}.$$

The average propensity to consume declines as income increases. The average propensity to consume is greater than the marginal propensity to consume, by the amount of a/Y. The ratio of saving to income is termed the average propensity to save (APS), i.e.

$$APS \equiv \frac{Y - C}{Y} = 1 - b - \frac{a}{Y}.$$

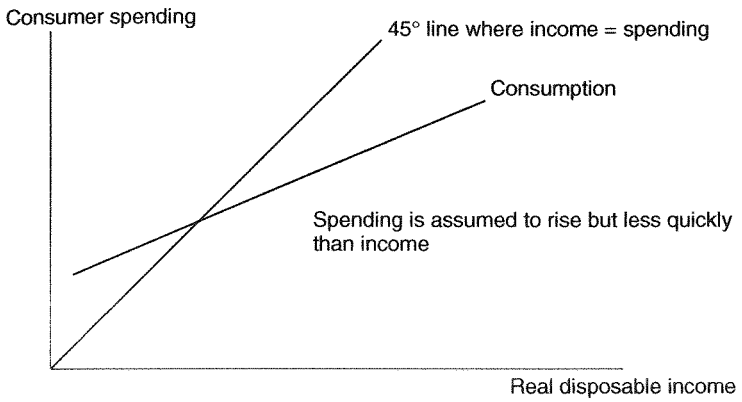


Figure 1.5.1 The Keynesian consumption function.

We have

$$APC + APS = 1.$$

As Y increases, the average propensity to save rises.

The Keynesian consumption function is based on the assumption that consumption reacts only to actual levels of income. It is known that the permanent income hypothesis would relax the Keynesian assumption. We now show that the Keynesian assumption is related to but different from the implications of our rational choice theory with consumption and saving. One of the implications of the Keynesian assumption is that the intercept a is independent of any change in wealth and other factors. Consider two persons, A and B, who started to work 5 years ago with the same conditions and the same preference. Their preferences were invariant during the period. Person A inherits a large amount of wealth at the beginning of the period (also assume zero interest income from the inherited wealth); nevertheless, person B accumulates little from work. At the end of the period, they both lose the job. If the interest rate is almost zero, both persons A and B have no income. According to the Keynesian consumption, at this moment when they lose their job or are retired, persons A and B should have the same level of consumption (which is given by the intercept a). Nevertheless, intuitively this is invalid as person A is richer than person B. It is reasonable to see that person A consumes more than person B. Our theory will solve this problem.

The OSG model with the Cobb-Douglas utility function determines the relationship between consumption and disposable personal income

$$C(t) = \xi \hat{Y}(t) = \xi Y(t) + \xi K(t).$$

We now connect the two theories by treating a in the Keynesian model as a wealth-related variable. If we assume that the intercept a is dependent on wealth

and marginal propensity to propensity, b, is related to the propensity to consume, ξ, in the following way

$$a = \xi K(t), \quad b = \xi,$$

then the Keynesian consumption function is identical to the consumption function in the OSG model. We may call our consumption function as a generalized Keynesian consumption function. We can define the APC and APS, denoted by $\bar{c}(t)$ and $\bar{s}(t)$, respectively, for the OSG model in the same way as in the Keynesian consumption function. In the OSG model

$$\bar{c}(t) = APC \equiv \frac{C}{Y} = \frac{\xi(Y+K)}{Y} = \xi + \xi\frac{K}{Y},$$

$$\bar{s}(t) = APS \equiv \frac{S-K}{Y} = \frac{\lambda Y - \delta K}{Y} = \lambda - \xi\frac{K}{Y},$$

where we use $\xi + \lambda = 1$. It should be noted that according to the definition of the APS

$$\bar{s}(t) = APS \equiv \frac{S(t)-K(t)}{Y(t)} = \frac{\dot{K}(t)}{Y(t)}.$$

The APC in the OSG model rise as wealth increases or as current income declines; the APS in the OSG model rise as wealth falls or as current income rises. It should be noted that $APC + APS = 1$. It is possible for the APS to be negative in the OSG model.

It should be noted that the life cycle hypothesis has also played an important role in analyzing households' behavior. The hypothesis was developed by Irving Fisher and Roy Harrod, before later being extended by Albert Ando and Franco Modigliani. It assumes that individuals consume a constant percentage of the present value of their life income. This is dictated by preference and income. The hypothesis is to explain the empirical work on consumption function.[24] It has been observed that the relationship between consumption and current income would be nonproportional and the intercept of consumption function is not constant over time. As stated by Modigliani,[25]

> The point of departure of the life cycle model is the hypothesis that consumption and saving decisions of households at each point of time reflect a more or less conscious attempt at achieving the preferred distribution of consumption over the life cycle, subject to the constraint imposed by the resources accruing to the household over its lifetime.

Consumption depends not just on current income but also on long-term expected earnings over their lifetime. Zhang (2005a) shows how the OSG model can be related to this hypothesis.[26]

1.6 The Solow growth model and the OSG approach

The Solow growth model, often also called the neoclassical growth model, is a work horse of economic growth theory. The model attempts to explain the long run economic development. A main problem for the model is that it lacks rational mechanism for explaining household behavior. When faced with choice of multiple goods (like leisure and services), the Solow model does not provide a mechanism for households to make decision. Many extensions of the Solow model in the 1960s and 1970s have limited implications because of the lacking of rational mechanism for household behavior. In his Nobel Lecture in 1987, Solow stated what he felt about the traditional growth theory:[27]

> Growth theory, like much else in macroeconomics, was a product of the depression of the 1930s and of the war that finally ended it. So was I. Nevertheless, it seemed to me that the story told by these models felt wrong. An expedition from Mars arriving on Earth having read this literature would have expected to find only the wreckage of a capitalism that had shaken itself to pieces long ago. Economic history was indeed a record of fluctuations as well as of growth, but most business cycles seemed to be self-limiting. Sustained, though disturbed, growth was not a rarity.

Most neoclassical models are extensions and generalizations of the pioneering works of Solow and Swan in 1956.[28]

The behavior of the production sector is the same as in the OSG model in Section 1.2.[29] Nevertheless, the Solow model assumes that the agents regularly set aside some fairly predictable portion $\hat{s}$ of its output for the purpose of capital accumulation; hence

$$\dot{k}(t) = \hat{s}f(k(t)) - (n + \delta_k)k(t).$$

We see that the differential equation for per-worker-capital accumulation in the Solow model is mathematically identical to the capital accumulation equation in the OSG model defined by equation (1.1.12) in Section 1.2

$$\dot{k}(t) = \lambda f(k(t)) - (\xi + n)k(t),$$

if we specify $U = c^{\xi}s^{\lambda}$. The Solow model and the OSG model have the same dynamic properties – the system has a unique stable equilibrium. But the OSG model holds that the saving rate is time-dependent; the Solow model predetermines the saving rate.

We now show that under certain circumstances the OSG model can explain what the Solow model forecasts. The OSG model endogenously determines saving and consumption. For simplicity, we let $\delta_k = 0$. The OSG's capital accumulation is given

$$\dot{K}(t) = \lambda F(t) - \xi K(t). \tag{1.6.1}$$

In the OSG model, the APS is given by

$$\bar{s}(t) = \frac{\lambda F(t) + \lambda K(t) - K(t)}{F(t)}.$$

We are interested in when $\bar{s}(t)$ in the OSG model equals the predetermined saving rate $\hat{s}$ in the Solow model, i.e.

$$\frac{\lambda F(t) + \lambda \delta K(t) - \delta K(t)}{F(t)} = \hat{s}. \tag{1.6.2}$$

If the propensity to save λ is considered as an endogenous variable, the above equation holds if

$$\lambda(t) = \hat{s} + (1-\hat{s})\delta \frac{K(t)}{F(t) + \delta K(t)} = \hat{s} + \frac{(1-\hat{s})\delta}{f(k(t))/k(t) + \delta} (< 1). \tag{1.6.3}$$

As $\xi(t) + \lambda(t) = 1$, $\xi(t)$ is also a function of $k/f(k)$. If the propensity to save is related to the ratio of capital per capita and output per capita as in equation (1.6.3), then $\bar{s}(t)$ in the OSG model is constant and is equal to the saving rate, $\hat{s}$, in the Solow model. Inserting equation (1.6.3) into equation (1.6.1) yields

$$\dot{K}(t) = \lambda(t)F(t) - (\xi(t) + \lambda(t)\delta_k)K(t) = \lambda(t)(F(t) + \delta K(t)) - K(t)$$
$$\times \hat{s}F(t) - \delta_k K(t).$$

We see that under equation (1.6.3) the evolution of capital in the OSG model is identical to that in the Solow model.

Theorem 1.6.1
Let the production sectors be identical in the OSG model and the Solow model. If the saving rate, $\hat{s}$, in the Solow model and the propensity to save $\lambda(t)$ in the OSG model satisfy equation (1.6.3), then the OSG model is identical to the Solow model in terms of the saving rate (out of current income), the consumption rate, the interest rate, the wage rate, output, income, consumption, and saving.

The propensity to save, $\lambda(t)$, is not constant. To see how it changes as economic conditions vary, we take the derivative of equation (1.6.3) with respect to t

$$\dot{\lambda} = \frac{(1-\hat{s})(f - kf')}{(f+k)^2}\dot{k} = \frac{(1-\hat{s})w}{(f+k)^2}\dot{k}.$$

The propensity to save rises (falls) as wealth per capita rises (falls).

1.7 The Ramsey growth model and the OSG approach

The Ramsey growth model is a neoclassical model of economic growth based primarily on the work of Frank Ramsey. Keynes considered Ramsey's 1928 classic

paper to be "one of the most remarkable contributions to mathematical economics ever made."[30]

The influences of Ramsey's classical contribution are reflected in the fact that almost all the contemporary dynamic models of national, urban, interregional, or international economic growth with microeconomic foundation for behavior of households are based on the paper and its variations (like the overlapping-generations model in discrete version).[31]

Most aspects of the Ramsey model are similar to the OSG model defined in Section 1.2. The variables, $F(t), K(t), N(t), k(t), w(t), r(t)$, in the Ramsey model have the same meanings as in the OSG model. The production process, marginal conditions, and population growth are the same as in the OSG model. We now describe households' behavior. In the Ramsey approach, households' decisions on saving are represented by assuming that consumers maximize the discounted value of their flow of utility, using a constant rate of impatience. The extended family is assumed to grow at an exogenously given rate n. Let the number of adults at time 0 be unity, the family size at time t is $N(t) = e^{nt}$. Each member supplies one unit of labor per unit time, without disutility. The household's preferences are expressed by an instantaneous utility function, $u(c(t))$, where $c(t)$ is the flow of consumption per person, and a discount rate for utility, denoted by ρ. For simplicity, specify $u(c)$ as

$$u(t) = \frac{c^{1-\theta}(t) - 1}{1-\theta}, \quad \theta > 0, \tag{1.7.1}$$

where θ is a parameter, $u' > 0, u'' < 0$, and u satisfies the Inada conditions: $u' \to \infty$ as $c \to 0$ and $u' \to 0$ as $c \to \infty$.

Assume that each household maximizes utility U as given by

$$U = \int_0^\infty u(c(t))e^{nt}e^{-\rho t}dt, \quad c(t) \geq 0, \quad t \geq 0.$$

The household makes the decision subject to a lifetime budget constraint. We denote the net assets per household by $k(t)$ which is measured in units of consumables. The total income at each point of time is equal to $w + rk$. The flow budget constraint for the household is

$$\dot{k} = w + rk - c - nk = f(k) - c - nk. \tag{1.7.2}$$

The equation means that the change rate of assets per person is equal to per capita income minus per capita consumption and the term, nk. It is assumed that the credit market imposes a constraint of borrowing, the present value of assets must be asymptotically nonnegative, that is

$$\lim_{t\to\infty}\left[k(t)\exp\left\{-\int_0^t (\rho - n)dv\right\}\right] \geq 0.$$

The present-value Hamiltonian is given by

$$J = u(c)e^{-(\rho-n)t} + \bar{\lambda}(w + rk - c - nk),$$

where $\bar{\lambda}$ is the present-value shadow price of income. The first-order conditions are

$$\frac{\partial J}{\partial c} = 0 \Rightarrow \bar{\lambda} = u' e^{-(\rho-n)t},$$

$$\frac{d\lambda}{dt} = -\frac{\partial J}{\partial k} \Rightarrow \frac{d\lambda}{dt} = -(\rho - n)\bar{\lambda}. \tag{1.7.3}$$

The transversality condition is given by

$$\lim_{t\to\infty} [\lambda(t)k(t)] = 0.$$

By equation (1.7.3), we can derive

$$r = \rho - \frac{u''c}{u'}\left(\frac{1}{c}\frac{dc}{dt}\right). \tag{1.7.4}$$

This equation implies that households will choose consumption so as to equate the rate of return r to the rate of time preference ρ plus the rate of decrease of the marginal utility of consumption u' due to growing per capita consumption c. Inserting equation (1.7.1) in equation (1.7.4) yields

$$\dot{c} = \frac{r-\rho}{\theta}c(t) = \frac{f' - \rho}{\theta}c. \tag{1.7.5}$$

The trajectory of the economy is determined by equations (1.7.2) and (1.7.5). The phase diagram in $c(t)$ and $k(t)$ is shown by Figure 1.7.1. Along the vertical line defined by $f'(k^*) = \rho$, the change rate of consumption per capita is equal to zero, that is, $\dot{c} = 0$. The consumption per capita increases to the left of the curve and falls to the right. Along the locus defined by

$$c = f - nk,$$

the change rate of the capital–labor ratio is equal to zero. We see that the capital–labor ratio falls above the curve and increases below it. With the requirement $\rho > n$ (without which the utility becomes unbounded along feasible paths), the intersection of the two curves determines a unique steady state, (k^*, c^*).

The local stability of the c–k system is determined by the characteristic roots of the following matrix of the coefficients of $k(t)$ and $c(t)$ equations linearized around the equilibrium point

$$J \equiv \begin{bmatrix} \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial c} \\ \frac{\partial \dot{c}}{\partial k} & \frac{\partial \dot{c}}{\partial c} \end{bmatrix} = \begin{bmatrix} f' - n & -1 \\ \frac{f''c}{\theta} & 0 \end{bmatrix}.$$

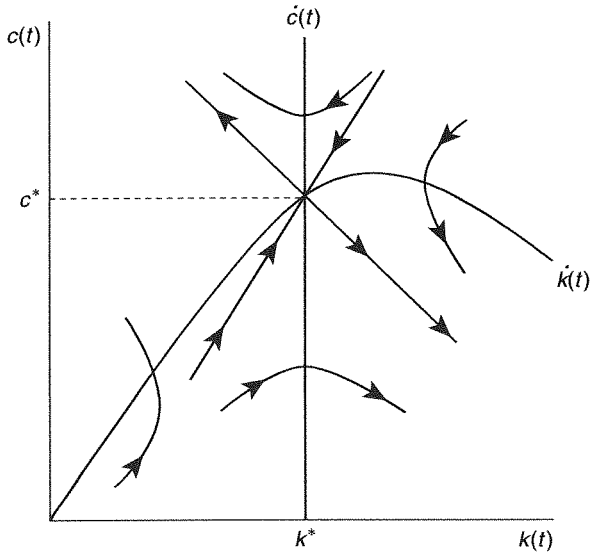


Figure 1.7.1 The dynamics of the Ramsey model.

We have

$$\mathrm{tr}(J) = f' - n = \rho - n > 0, \quad |J| = \frac{f''c}{\theta} < 0. \tag{1.7.6}$$

The characteristic equation is

$$\phi^2 - \mathrm{tr}(J)\phi + |J| = 0.$$

Inserting equation (1.7.6) in the above equation yields

$$\phi^2 - (\rho - n)\phi + \frac{f''c}{\theta} = 0 \Rightarrow \phi_{1,2} = \frac{(\rho - n) \pm \sqrt{(\rho - n)^2 - 4f''c/\theta}}{2}.$$

The Ramsey model is controlled by a system of two differential equations. Together with the initial conditions and the transversality condition, this system determines the path of the two variables. At stationary state, the per capita variables, k, c, and $y(\equiv Y/N)$, grow at the rate 0, and the level variables, K, C, and Y, grow at the rate, n. It can be shown that the system has a unique steady state. Since the two eigenvalues have the opposite signs, the system is locally saddle-path stable.[32]

The dynamical behavior of the Ramsey model is controlled by

$$\dot{k} = w + rk - c - nk = f - c - nk,$$

$$\dot{c} = \frac{r - \rho}{\theta}c = \frac{f'(k) - \rho}{\theta}c. \tag{1.7.7}$$

If we find some equation of preference change in the OSG model to generate the same behavior as equation (1.7.7), then the two systems should exhibit the same behavior in terms of consumption, capital accumulation and incomes, even though they are built on different assumptions.

We now consider consumption of the OSG model. The consumption per capita in the OSG model is given by

$$c(t) = (1 - \lambda(t))[f(k(t)) + k(t)].$$

Differentiation of this equation with respect to time yields

$$\frac{\dot{c}}{c} = \frac{f' + \delta}{f(k) + \delta k}\dot{k} - \frac{\dot{\lambda}}{1 - \lambda}. \tag{1.7.8}$$

For equations (1.7.7) and (1.7.8) to be equal, it is sufficient for $\lambda(t)$ to evolve according to

$$\dot{\lambda} = \frac{f' + 1}{f + k}\xi\dot{k} - \frac{f' - \rho}{\theta}\xi. \tag{1.7.9}$$

The propensity to own wealth λ tends to rise (fall) when $\dot{k}$ rises (falls); it tends to rise (fall) when $r < (>)\rho$. We may interpret that the direction of change in λ is influenced by the direction of change in wealth as well as whether the rate of return of wealth is larger or smaller than the rate of time preference. If the wealth is increasing and the rate of time preference is larger than the rate of return, then the propensity to save will definitely rise. If the wealth is falling and the rate of time preference is smaller than the rate of return, the propensity tends to fall. In the other cases, the propensity may either increase or decrease.

Under equation (1.7.9), the consumption per capita in the OSG model evolves in the same way as in the Ramsey model. We now examine the fundamental equation of the OSG, i.e.

$$\dot{k} = \lambda f(k) - (1 - \lambda + n)k.$$

By $c = (1 - \lambda)(f + k)$, the above equation can be rewritten as

$$\dot{k} = f(k) - c - nk. \tag{1.7.10}$$

Theorem 1.7.1
Let the production sectors be identical in the OSG model and the Ramsey model. If the propensity to save, $\lambda(t)$, evolves according to equation (1.7.9), then the OSG model generates the same dynamics of capital–labor ratio, $k(t)$, and per-capita consumption, $c(t)$, as the Ramsey model does.

This example illustrates how the Ramsey model is related to the OSG model. We can similarly examine relationships between the two approaches when utility functions are taken on other forms.

We now explain another difference between the Solow and Ramsey approaches. The Solow model determines consumption as follows

$$c(t) = (1 - \hat{s})f(k(t)).$$

Taking the derivatives of this relation with respect to t yields

$$\dot{c}(t) = (1 - \hat{s})f'(k)\dot{k}(t).$$

The change rate of consumption is positively related to the rate of interest. The rational household has increasing, stationary, or decreasing consumption according to whether the wealth rises, is stationary, or falls. The consumer adapts consumption level not according to the difference between the interest rate and discount rate for utility as the Ramsey model predicts. According to the Solow model theory, a Japanese consumer would consume more, irrespective of low interest, if his wealth increases; he would consume less, irrespective of a high interest rate, when his wealth falls. On the other hand, the Ramsey model predicts

$$\dot{c}(t) = \frac{f'(k(t)) - \rho}{\theta} c(t).$$

This implies that the difference between r and ρ determines whether households choose a pattern of per capita consumption that rises, stays constant or falls over time. The optimizing household has increasing, stationary, or decreasing consumption according as the current real interest rate exceeds, equals, or falls short of the utility discount rate. According to this result, consumption always falls if the interest rate is low and the utility discount rate is high.

It should be noted that in 1937 Paul Samuelson published an article on discounted utility. Since then, the discounted utility was rapidly adopted as the framework of choice for intertemporal decisions. It is worthwhile to cite from Samuelson's following cautions:

> Any connection between utility as discussed here and any welfare concept is disavowed.[33]
>
> It is completely arbitrary to assume that the individual behaves so as to maximize an integral of the form envisaged in [the discounted utility model].[34]

It is easy to see what Samuelson means by examining the form of utility formulation in the Ramsey optimal growth theory

$$\int_0^\infty U[C(t)]e^{-\rho t}dt.$$

The specified form means that the household's utility at time 0 is a weighted sum of all future flows of utility. The parameter, $\rho(\geq 0)$, is defined as the rate of

time preference. A positive value of ρ means that utilities are valued less the later they are received. There are two assumptions involved in the Ramsey model. The first is that utility is additional over time. Although we may add capital over time, it is unrealistic to add utility over infinite time. Intuitively it is not reasonable to add happiness over time. It is well known in utility theory that when we use utility function to describe consumer behavior an arbitrary increasing transformation of the function would result in identical maximization of the consumer at each point of time. Obviously, the above formulation will not result in an identical behavior if U is subjected to arbitrarily different increasing transformations at different times. The second implication of the above formation is that the parameter ρ is meaningless if utility is not additional over times. It should be noted that Ramsey considered the meanings of this parameter from ethical perspectives. Ramsey interpreted the agent as a social planner, rather than a household. The planner chose consumption and saving for the current and future generations. Ramsey assumed $\rho = 0$ and considered $\rho > 0$ "ethically indefensible." If $\rho = 0$, by equation consumption per capita always grows if the interest rate is positive irrespective of whether wealth grows or falls.

The Ramsey framework has not been proved to be effective when we take account of variations in households' preferences. As empirical studies have convinced existence of great differences in impatience among households and the Ramsey approach proves futile for dealing with the issues, it is necessary to search alternative ways to explain the reality. In fact, as shown in a recent survey on studies of estimating individuals' discount rates by Frederick *et al.*[35], rates differ dramatically across studies, and within studies across individuals. There is no convergence toward an agreed-on rate of impatience. It is estimated, for instance, by Warner and Pleeter[36] that individual discount rates vary between 0 and 70 percent per year. Frederick *et al.* observe

> The [discounted utility] model, which continues to be widely used by economists, has little empirical support. Even its developers – Samuelson, who originally proposed the model, and Koopmans, who provided the first axiomatic derivation – had concerns about its descriptive realism, and it was never empirically validated as the appropriate model for intertemporal choice.
>
> ... [D]eveloping descriptively adequate models of intertemporal choice will not be easy.[37]

Although the validity of the Ramsey approach has been questioned over years, it is quite another matter to create a "more effective" alternative.[38] Instead of searching for another approach, the main attitude toward the approach is illustrated by Turnovsky (2000: 273).

> Any model as widely employed as the representative agent model begins to take on a life of its own and to be accepted almost as an axiom. It is therefore useful to remind ourselves periodically of its limitations. Despite the criticisms

> that have been made, we feel that the representative agent model provides a useful framework that offers a good deal of insight, and we shall continue to develop it further.

This "reformist" attitude of refinement and extensions will not solve essential problems in the approach.[39]

This book uses an alternative utility function, not only because the validity of the discounted utility concept has been questioned from philosophical, psychological, or/and empirical aspects,[40] but also because many obviously significant issues, such as growth with heterogeneous households and growth with interregional dynamics, cannot be properly discussed with the concept as having become evident in the history of theoretical economics in the last 40 years. Since the Ramsey approach has been so influential in the contemporary monetary growth theory, monetary growth models are described within the approach in the main context.

1.8 The structure of the book

Contemporary economic theory includes a number of separate sub-theories. It is more desirable to have one compact theoretical framework than multiple separate theories to explain various aspects of the reality. I have made efforts to examine various aspects of national economies with a few concepts within a consistent framework.[41] This book explores interdependence between economic growth and money. The book is focused on monetary growth theory.[42] It is for graduate students and researchers in monetary economics, macroeconomics and economic growth theory. Applied mathematicians may also find some interesting problems for applying technical skills to solve, for instance, nonlinear dynamic problems. Some of the models are linked to the calibration and simulation techniques commonly used in dynamic analysis. The use of dynamic simulations to follow processes of economic evolution is important, as most of the nonlinear dynamic models cannot be explicitly solved. The book is organized as follows.

Chapter 2 is organized as follows. Section 2.1 introduces the Tobin model based on the work by Levhari and Patinkin. Section 2.2 develops a monetary growth model which is similar to the Tobin model in many aspects. The main difference between this model and the Tobin is how to model the household choice of saving and consumption. Section 2.3 extends the model in Section 2.2 by including bonds in the monetary growth model. Section 2.4 extends the model in Section 2.2 to deal with issues related to income and wealth distribution in a monetary growth model. We are concerned with a two-group growth model. Section 2.5 extends the model in Section 2.2 to deal with issues related to economic structural change in a monetary growth model. The section proposes a two-sector growth model with money. Section 2.6 discusses issues related to money in growth models. Appendix 2.1 introduces the generalized Tobin model. Appendix A. 2.2 demonstrates the existence economic chaos in a simple monetary economy.

Chapter 3 proposes some monetary growth models with the money-in-utility function (MIUF) approach. Section 3.1 introduces the MIUF growth model by Levhari and Patinkin. Section 3.2 examines the MIUF model by Sidrauski and its contemporary variations. Section 3.3 develops the one-sector MIUF-OSG model. Section 3.4 extends the one-sector MIUF-OSG model to a model with two types of households. Section 3.5 extends the model in Section 3.3 by including bonds in the monetary growth model. In Section 3.6, we develop a growth model with both human and capital accumulation. Section 3.7 discusses a growth model with the money in production function approach. In Section 3.8, we mention a few other issues related to the MIUF. Appendix 3.1 proves the results in Section 3.6. Appendix 3.2 introduces a MIUF model proposed by Matsuyama. The example demonstrates how chaos occurs in a simple monetary model. Appendix 3.3 extends the one-sector MIUF-OSG model to the case with many types of households. Appendix 3.4 extends the one-sector MIUF-OSG model to take account of leisure in our model.

Chapter 4 studies the money-in-advance (CIA) approach. Section 4.1 introduces money into the OSG growth model with the CIA approach. Section 4.2 generalizes the model in Section 4.1 by assuming that the economy consists of two production sectors. Section 4.3 deals with a monetary model with the CIA approach in which the monetary policy takes the form of Taylor's interest rate feedback rule. Section 4.4 introduces home production and durable goods into the model in Section 4.1. Section 4.5 deals with interdependence between economic growth and knowledge growth through learning by doing and research in a monetary economy with the CIA approach. Section 4.6 introduces public goods into the model in Section 4.1. Section 4.7 introduces a shopping cost model in the Ramsey approach. Section 4.8 concludes the chapter. The Appendix discusses a monetary growth economy with two types of households with the CIA approach.

Chapter 5 is organized as follows. Section 5.1 introduces a traditional model of inflation and unemployment. Section 5.2 studies the well-known IS-LM model of a national economy. In Section 5.3, we propose a monetary growth model with unemployment and welfare policy in the MIUF approach. Section 5.4 synthesizes the wage efficiency theory and the OSG-MIUF model. In Section 5.5, we discuss some other causes of unemployment. Appendix 5.1 introduces a simple model of unemployment and inflation. The model exhibits chaos.

Chapter 6 deals with issues related to preference change and habit persistence in monetary growth models. Section 6.1 introduces money into the Uzawa model with preference change. Section 6.2 proposes a CIA-OSG model with preference change. Section 6.3 proposes a MIUF-OSG model with preference change. Section 6.4 develops a two-sector two-group model with preference change and the MIUF approach. Section 6.5 introduces habit formation in the Ramsey approach. Appendix A.6.1 studies a two-sector multi-group non-monetary model with general production functions.

The objective of Chapter 7 is to study monetary growth with economic geography. We are concerned with the residential land use pattern and determination of goods production, capital accumulation, housing price, land rent over time

and space. The model is a one-dimensional model of residential location with a central business center. Section 7.1 defines the basic model. Section 7.2 shows how to solve the dynamics with economic geography, examines the motion of some variables over space, and studies the dynamic properties of the model. Section 7.3 applies the standard comparative statics analysis to examine the impact of changes in some parameters. Section 7.4 demonstrates the existence of a unique equilibrium point, calculates the equilibrium values of all the variables. Section 7.5 simulates the motion of the dynamic system over space and carries out comparative dynamic analysis with regard to the propensity to save and the propensity to consume housing. Section 7.6 concludes the chapter. The appendix proves the main results in Section 7.2.

Chapter 8 proposes a two-region growth model with money. Section 8.1 defines the two-region monetary growth model with capital accumulation. Section 8.2 shows that the dynamics of the non-monetary national economy can be described by 2-dimensional differential equations and demonstrates existence of a unique equilibrium point of the dynamic system. This section also examines the effects of changes in different regions' amenity parameters, productivity levels and residential areas, and the propensity to save upon the national economic growth and regional economic structures. Section 8.4 shows that the monetary economic system can be represented by 3-dimensional differential equations. The section also provides a procedure to determine all the variables at any point of time. Section 8.5 concludes the chapter.

Chapter 9 is organized as follows. Section 9.1 introduces a classical model of open monetary economy with the gold standard. Section 9.2 studies Dornbush's exchange rate model for a small economy. Section 9.3 proposes a small open-country monetary with the CIA approach. Section 9.4 proposes a small open-country monetary with the MIUF approach. Section 9.5 builds a world monetary economic model with the MIUF approach. Section 9.6 concludes the chapter. Appendix A.9.1 introduces the IS-LM model for an open economy.

Chapter 10 concludes the study by pointing out important issues for future study within the framework proposed in this study.

APPENDIX

A.1.1 The OSG model in discrete time

The neoclassical growth theory is developed both in continuous and discrete versions. In his original contribution, Diamond (1965a) used the overlapping generations structure as proposed by Samuelson (1958) to examine the long-term dynamical efficiency of competitive production economies. The model has become a standard tool in macroeconomic to study economic dynamics in general. Similarly, our alternative approach is developed both in continuous and discrete versions. Because of the character of our alternative utility function, it is straightforward to change discrete models to continuous models, and vice versa.[43]

The continuous version requires knowledge about differential equations; while the discrete version requires knowledge about difference equations. It should be noted that, as mentioned by Bunzel (2006), a large part of the interest in the dynamics of OLG models can be attributed to the fact that these models more easily generate endogenous complicated dynamical behavior (such as bifurcations, periodic and aperiodic solutions, and chaos) than models with representative infinitely lived agents.

We now represent the OSG model in discrete time. The economy has an infinite future. We represent the passage of time in a sequence of periods, numbered from zero and indexed by $t = 0, 1, 2, \ldots$. Time 0, being referred to the beginning of period 0, represents the initial situation from which economy starts to grow. The end of period $t-1$ coincides with the beginning of period t; it can also be called time t. We assume that transactions are made in each period. The model assumes that each individual lives forever. As before, population grows at rate n; thus

$$N_t = (1+n)N_{t-1}.$$

Each individual supplies one unit of labor at each time t. Production in period t uses inputs amount of capital, K_t, and amount of labor services, N_t. It supplies amount Y_t of goods. Here, production is assumed to be continuous during the period, but then use the same capital that existed at the beginning of the period.

The production function is $F_t = F(K_t, N_t)$. The marginal conditions are

$$r_t + \delta_k = f'(k_t), \quad w_t = f(k_t) - k_t f'(k_t), \tag{A.1.1.1}$$

where

$$k_t \equiv \frac{K_t}{N_t}, \quad f_t \equiv \frac{F_t}{N_t} = F(k_t, 1).$$

As in the OSG model in continuous time, the consumer is to choose his most preferred bundle (c_t, s_t) of consumption and saving under his budget constraint. The utility maximizing problem at any time is defined by

$$\underset{c_t, s_t \geq 0}{Max}\ U(c_t, s_t)$$

$$\text{s.t. } c_t + s_t = \hat{y}_t \equiv f(k_t) + k_t.$$

The optimization problem is the same as the model in continuous time. Here, we will not repeat the proof. We denote an optimal solution as function of the disposable income

$$(c_t, s_t) = (c(\hat{y}_t), s(\hat{y}_t)).$$

The wealth accumulation is given by

$$K_{t+1} - K_t = s(\hat{y}_t)N_t - K_t.$$

That is

$$K_{t+1} = s(\hat{y}_t)N_t.$$

Dividing the two sides of this equation by N_{t+1} yields

$$k_{t+1} = \frac{s(\hat{y}_t)}{1+n}.$$

This mapping controls the motion of the system with general production and utility functions. A stationary state for the growth progress is a capital–labor ratio, k^*, that satisfies

$$k = \frac{s(\hat{y})}{1+n}. \tag{A.1.1.2}$$

To solve the above equation, define

$$\Phi(k) \equiv \frac{s(\hat{y})}{(1+n)k} - 1, \quad k \geq 0. \tag{A.1.1.3}$$

When k is approaching zero, $\hat{y}(=f(k)+k)$ is also approaching zero, and hence $s(\hat{y})$ is coming near zero. As $0 < s'(\hat{y}) < 1$ and $f'(k) \to \infty$ as $k \to 0$

$$\lim_{k\to 0} \frac{s(\hat{y})}{(1+n)k} = \frac{s'(0)(f'(0)+1)}{(1+n)} > 1. \tag{A.1.1.4}$$

When k is approaching positive infinity, $\hat{y}$ is coming to positive infinity. As $0 < s'(\hat{y}) < 1$ and $f'(k) \to 0$ as $k \to +\infty$, we have

$$\lim_{k\to+\infty} \frac{s(\hat{y})}{(1+n)k} = \frac{s'(+\infty)(f'(+\infty)+1)}{(1+n)} < 1. \tag{A.1.1.5}$$

Taking derivatives of equation (A.1.1.3) with respect to k yields

$$\frac{d\Phi}{dk} = \left[\frac{s'(\hat{y})k(f'(k)+1)}{s(\hat{y})} - 1\right]\frac{s(\hat{y})}{(1+n)k^2}.$$

We now show that $d\Phi/dk < 0$ for $k > 0$. To prove this, we use $ds/d\hat{y} < 1$ and the inequality $f'(k) < f(k)/k$ (which also guarantees $w > 0$). By equation (A.1.1.1) and the definition of $\hat{y}$, we have

$$\frac{s'(\hat{y})k(f'(k)+1)}{(1+n)s(\hat{y})} < \frac{s'(\hat{y})(f(k)+k)}{(1+n)s(\hat{y})} = \frac{s'(\hat{y})\hat{y}}{(1+n)s(\hat{y})} < 1,$$

where we use $s'(\hat{y}) < 1$ and $\hat{y}/s(\hat{y}) \leq 1$ to guarantee the right inequality. We thus conclude $d\Phi/dk < 0$ for $k > 0$. The equation, $\Phi(k) = 0$ for $k > 0$, has a

unique solution because of equations (A.1.1.4), (A.1.1.5), and $d\Phi/dk < 0$. We now demonstrate that the unique stationary state is stable.

For the steady state to be stable, it is sufficient to show

$$-1 < \frac{s'(\hat{y}^*)(f'(k^*)+1)}{1+n} < 1.$$

As the steady-state values, k^* and $\hat{y}^*$ are positive, $s'(\hat{y}^*) > 0$ and $f'(k^*)$ are positive. The left inequality always holds. To prove the right inequality, we use $ds/d\hat{y} < 1$ and the inequality $f'(k^*) < f(k^*)/k^*$ (which also guarantees $w_t > 0$). By equation (A.1.1.1) and the definition of $\hat{y}$, we have

$$\frac{s'(\hat{y}^*)(f'(k^*)+1)}{1+n} < \frac{s'(\hat{y}^*)(f(k^*)/k^*+1)}{1+n} = \frac{s'(\hat{y}^*)\hat{y}^*/k^*}{1+n} = \frac{s'(\hat{y}^*)\hat{y}^*}{s(\hat{y}^*)}.$$

Since

$$s'(\hat{y}) < 1, \quad \frac{\hat{y}}{s(\hat{y})} \leq 1,$$

we see that the right inequality of inequalities (A.1.1.3) is satisfied.

Theorem A.1.1.1
Given a production function that is neoclassical and a utility function that is a C^2 function, and satisfies $U_c > 0, U_s > 0$ for any $(c_t, s_t) > 0$. Let the bordered Hessian be positive for any nonnegative $(c_t, s_t) > 0$. Then the capital–labor ratio converges monotonically to a unique positive steady state. The unique stationary state is stable.

We conclude that the unique steady state is stable. Figure A.1.1 shows the relation between k_{t+1} and k_t, which we express by

$$k_{t+1} = \Psi(k_t).$$

The slope of $\Psi(k_t)$ is infinite at $k_t = 0$ and diminishes toward a constant. The function $\Psi(k_t)$ crosses the 45-degree line at the steady-state value, k^*. The capital stock monotonically approaches its unique steady-state value as time passes. The steady state is stable because the curve $\Psi(k_t)$ is always upward sloping, and it crosses the 45-degree line from above.

A.1.2 The overlapping-generations (OLG) model and its relations to the OSG model

The overlapping-generations model is another key modeling framework in economic dynamics. This approach is basically the discrete version of the Ramsey model. Since the approach allows people to live in two periods,[44] many economic

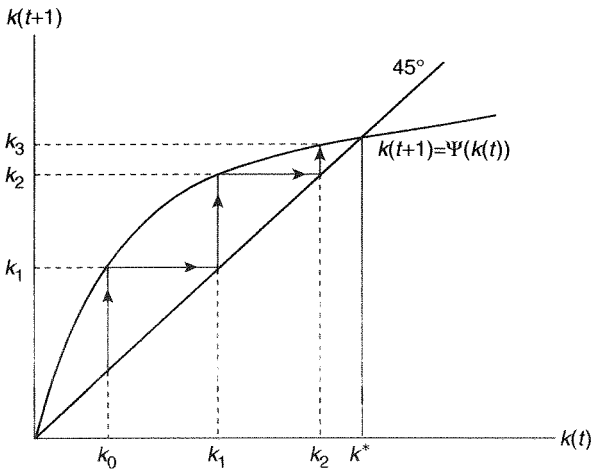


Figure A.1.1 Dynamics in the OSG model.

problems related to physical capital accumulation or/and human capital can be constructed in such a way that the resulted models are analytically tractable. The tractability is perhaps the main reason for the popularity of the analytical framework.

This section introduces the model of finite-horizon households.[45] It is common to assume, in the OLG analytical framework, that each person lives for only two periods.[46] People work in the first period and retire in the second. If one thinks of reality, one period perhaps lasts over 30 years. We neglect any possibility of transfers from government or from members of other generations. People consume in both periods; they pay for consumption in the second period by saving in the first period. The cohort that is born at time t is referred to as generation t. Members of this generation are young in period t and old in period $t+1$. At each point in time, members of only two generations are alive. Each person maximizes lifetime utility, which depends on consumption in the two periods of life. It is assumed that people are born with no assets and do not care about events after their death; they are not altruistic toward their children, and therefore, do not provide bequests or other transfers to members of the next generation. Their lifetime utility is specified as

$$U(t) = \frac{c_1^{1-\theta}(t)-1}{1-\theta} + \left(\frac{1}{1+\rho}\right)\frac{c_2^{1-\theta}(t+1)-1}{1-\theta}, \qquad \text{(A.1.2.1)}$$

where $\theta > 0, \rho > 0, c_i(t)$ is consumption of generation $i, i = 1, 2$. Each individual supplies one unit of labor inelastically while young and receives the wage income $w(t)$; he does not work when old. Let s_t stand for the amount saved in period t.

The budget constraint for period t

$$c_1(t)+s(t)=w(t). \tag{A.1.2.2}$$

Let $r(t+1)$ denote the interest rate on one-period loans between periods t and $t+1$. In period $t+1$, the individuals consume the savings plus the accrued interest

$$c_2(t+1)=(1+r(t+1))s(t). \tag{A.1.2.3}$$

For each individual, $w(t)$ and $r(t+1)$ are exogenous; he chooses $c_1(t)$ and $s(t)$ (and, hence c_{2t+1}) subject to equations (A.1.2.2) and (A.1.2.3). Substitute equations (A.1.2.2) and (A.1.2.3) into the utility function to delete $c_1(t)$ and $c_2(t+1)$

$$U(t)=\frac{(w(t)-s(t))^{1-\theta}-1}{1-\theta}+\left(\frac{1}{1+\rho}\right)\frac{(1+r(t+1))^{1-\theta}s^{1-\theta}(t)-1}{1-\theta}.$$

The first-order condition with respect to $s(t)$ is

$$s^{-\theta}(t)(1+r(t+1))^{1-\theta}=(1+\rho)(w(t)-s(t))^{-\theta}, \tag{A.1.2.4}$$

which also implies, under equations (A.1.2.2) and (A.1.2.3)

$$\frac{c_2(t+1)}{c_1(t)}=\left(\frac{1+r(t+1)}{1+\rho}\right)^{1/\theta}, \tag{A.1.2.5}$$

Solve equations (A.1.2.4)

$$s(t)=\frac{w(t)}{\varphi(t+1)}, \tag{A.1.2.6}$$

where

$$\varphi(t+1)\equiv 1+\frac{(1+\rho)^{1/\theta}}{(1+r(t+1))^{(1-\theta)/\theta}}>1.$$

We obtain

$$s_r\equiv\frac{\partial s(t)}{\partial r(t+1)}=\left(\frac{1-\theta}{\theta}\right)\left(\frac{1+\rho}{1+r(t+1)}\right)\frac{s(t)}{\varphi(t+1)},$$

$$s_w\equiv\frac{\partial s(t)}{\partial w(t)}=\frac{1}{\varphi(t+1)}, \quad 0<s_w<1. \tag{A.1.2.7}$$

We see

$$s_r\begin{cases}>0 \text{ if } \theta<1,\\ =0 \text{ if } \theta=1,\\ <0 \text{ if } \theta>1.\end{cases}$$

Firms' behavior is the same as in the Solow model. They have the neoclassical production function, $y(t) = f(k(t))$, where

$$y(t) \equiv \frac{Y(t)}{N(t)}, \quad k(t) \equiv \frac{K(t)}{N(t)},$$

where $N(t)$ is the number of young people, $Y(t)$ is the total output, $K(t)$ is the total capital. The marginal conditions are

$$r(t) = f'(k(t)) - \delta_k, \quad w(t) = f(k(t)) - k(t)f'(k(t)), \tag{A.1.2.8}$$

where δ_k is the depreciation rate of capital.

In the closed economy, households' assets equal the capital stock. Aggregate net investment equals total income minus total consumption

$$K(t+1) - K(t) = w(t)N(t) + r(t)K(t) - c_1(t)N(t) - c_2(t)N(t-1). \tag{A.1.2.9}$$

Substituting equations (A.1.2.8) into equation (A.1.2.9), we get

$$K(t+1) - K(t) = F(K(t), N(t)) - C(t), \tag{A.1.2.10}$$

where

$$C(t) = c_1(t)N(t) + c_2(t)N(t-1).$$

From equations (A.1.2.9), (A.1.2.2), and (A.1.2.3), we obtain

$$K(t+1) = s(t)N(t) + (1 + r(t))(K(t) - s_{t-1}(t)N_{t-1}(t)).$$

Assume that the economy starts with the condition

$$c_2(1)N(0) = (1 + r(1))K(1),$$

which is equivalent to $K(2) = s(1)N(1)$ with equations (A.1.2.9) and (A.1.2.2). Hence, the above equation becomes

$$K(t+1) = s(t)N(t),$$

which means that the savings of the young equal the next period's capital stock. We can write this equation in per capita terms

$$k(t+1) = \frac{s(t)}{1+n}.$$

Substituting $s(t)$ in equation (A.1.2.6) into the above equation yields

$$k(t+1)\left[1+\frac{(1+\rho)^{1/\theta}}{(1+r(t+1))^{(1-\theta)/\theta}}\right]=\frac{w(t)}{(1+n)}, \tag{A.1.2.11}$$

in which we use the definition of $\varphi(t+1)$. In the case that utility is logarithmic ($\theta=1$), it can be shown that equation (A.1.2.11) becomes

$$k(t+1)=\frac{1}{(1+n)(2+\rho)}\left[f(k(t))-k(t)f'(k(t))\right]. \tag{A.1.2.12}$$

The system has a unique equilibrium and it is stable.

It can be seen that our model is similar to the OLG model as far as stationary states are concerned. A careful reader might have already noticed that it is easy to relate the discrete OSG model and the OLG model. For instance, if we consider that people live only two periods in the OSG model as in the OLG model, then the saving $s(t)$ plus its interest is equal to the consumption level of the second period. Then, our OSG model with certain form of the utility function generates the same dynamic behavior as the OLG model. Although the OSG model generates the same dynamic behavior as the OLG model, the OSG model does not employ the concept of discounted utility or discount preference rate.

2 Money as a store of value

Modern analysis of the long-term interaction of inflation and capital formation begins with Tobin's seminal contribution in 1965. Tobin (1965) deals with an isolated economy in which "outside money"[1] competes with real capital in the portfolios of agents within the framework of the Solow model. With regard to outside money holdings as net wealth, Tobin (1965: 676) points out:

> The community's wealth ... has two components: the real goods accumulated through past real investment and fiduciary or paper 'goods' manufactured by the government from thin air. Of course the non-human wealth of such a nation 'really' consists only of its tangible capital. But, as viewed by the inhabitants of the nation individually, wealth exceeds the tangible capital stock by the size of what we might term the fiduciary issue. This is an illusion, but only one of the many fallacies of composition which are basic to any economy or any society. The illusion can be maintained unimpaired so long as the society does not actually try to convert all of its paper wealth into goods.

In Tobin's approach, a monetary economy has a real sector exactly like that in the Solow growth model, so that the monetary nature of the model depends on how money is introduced into the model. A monetary economy is characterized by that prices are expressed in money, transactions require money, and financial wealth can be held in the form of money or financial instruments competing with money. In the Tobin model, money is a liability of the public sector. As a depositor of purchasing power, money can be held by private agents as an alternative form of wealth to physical capital stock. Different from a barter economy as described by the Solow model, the Tobin model involves a problem of deciding the optimal composition of wealth at every instant. This chapter addresses similar issues to those in the Tobin model. We examine the consequences of financing deficits entirely by printing currency. That is, using inflation to tax the holding of money within an alternative approach to consumer decision making.

2.1 The Tobin monetary growth model

Before developing an alternative approach to the Tobin model, we introduce the Tobin analysis revised by Levhari and Patinkin (1968). A general Tobin model is provided in Appendix A.1. As in the Solow model introduced in Chapter 1, the production function is given by $Y = F(K, N)$, where Y, K, and N respectively represent net output, capital and effective labor. Assume that labor grows at a fixed rate, n. We write this function in intensity terms $y = f(k)$, where $y \equiv Y/N, k \equiv K/N, f' > 0$, and $f'' < 0$. Tobin introduced a government into the Solow model. The government spends a fixed proportion of the national income and finances this expenditure either by levying taxes or by printing money. This is analytically equivalent to the assumption that the sole function of the government is to issue paper money, which it injects into the economy by means of transfer payments – or withdraws by means of taxes. These payments (or taxes) are assumed independent of the agents' holdings of money. Assume that there are neither direct storage costs of holding money nor costs of administering the monetary system.

The model has two assets – physical capital and real money balances. According to Tobin, the disposable income is given by

$$Y_D = Y + \frac{d(M/P)}{dt} = Y + (\mu - \pi)\frac{M}{P},$$

where M is the quantity of money supplied, P the price level, $\mu (= \dot{M}/M)$ the rate of change of money, and $\pi (= \dot{P}/P)$ the rate of change of the price level. The real disposable income is equal to net income plus the real value of the increase in the nominal quantity of money less the decrease in the real value of existing cash balances caused by a price increase. Assume that the households save a certain proportion, s, of their disposable incomes and that they hold a certain proportion, ε, of the national income, Y, in the form of real money balances. We thus have

$$\dot{K} = Y - (1-s)Y_D = F - (1-s)\left[F + (\mu - \pi)\frac{M}{P}\right],$$

$$\frac{M^d}{P} = \varepsilon F(K, N).$$

The rate of return on physical capital is $r = f'(k)$ and the rate of return on real money balances is $-\pi$.[2] The agents may hold a portfolio which consists of both of the two assets. It is assumed that the anticipated rates of return from the two assets are always equal to the existing rates. In view of the greater liquidity of money balances, equilibrium cannot prevail unless the yield from these balances is less than that on physical capital, i.e. $-\pi < r$. We assume that ε is inversely dependent on the anticipated alternative cost of holding real money balances, that is, on the difference cost between what could have been earned from the holding of a unit of physical capital and what will be earned from the holding of money

balances. As the anticipated and actual rates are equal, this cost is $r-(-\pi)=r+\pi$, or Fisher's money rate of interest. We express the assumption that ε is inversely dependent on the anticipated alternative cost of holding real money balances by

$$\varepsilon=\varepsilon(r+\pi), \quad \varepsilon'<0.$$

Assume that the demand for real money balances is always equal to the supply. With this assumption, we have closed the model. We have built the Tobin model. It is straightforward to show that the equilibrium value of the capital–labor ratio, k, is determined by the condition

$$s-(1-s)n\varepsilon=\frac{nk}{f(k)}.$$

It should be noted that in the Solow model, the equilibrium condition is $s=nk/f(k)$. As $d(k/f)/dk>0$ and $0<s<1$, we see that the equilibrium value of k is lower than that in the Solow model. It is also easy to show that an increase in the rate of inflation increases the equilibrium value of k by taking the derivatives of the above equation with respect to $\mu(=\pi+n)$

$$\frac{dk}{d\mu}=-\frac{(1-s)f^2\varepsilon'}{(f/k-f')k+(1-s)f^2f''\varepsilon'}>0.$$

This means that money is not neutral in the long run. This conclusion is the most often mentioned conclusion in monetary growth theory. As mentioned late on, some models predict similar effects and some other models deny this conclusion, predicting either superneutrality in the long run or the reverse effect.[3]

2.2 Growth with inflationary finance

The model in this section is similar to the Tobin model in many aspects. The main difference is how to model household choice of saving and consumption.[4] As discussed by Orphanides and Solow (1990), Tobin's result is robust with respect to a variety of different modeling strategies. As Tobin (1956) pointed out, the major contribution of his own analysis is that he extends Keynesian reasoning to a situation of the long run. Tobin (1956: 684) explains

> In classical theory, the interest rate and the capital intensity of the economy are determined by 'productivity and thrift', that is, by the interaction of technology and saving propensities ... Keynes gave reasons why in the short run monetary factors and portfolio decisions modify, and in some circumstances dominate, the determination of the interest rate and the process of capital accumulation. I have tried to show here that a similar proposition is true for the long run.

This section shows that the Tobin effect prevails with the approach introduced in Chapter 1.

2.2.1 The model

The production sector is the same as in the OSG model described in Section 1.2. The production function is, $F(K(t), N)$. Output per capita is a function of capital intensity, $y(t) = f(k(t))$, where $k(t) \equiv K(t)/N$. The function f has the following properties: (i) $f(0) = 0$ (ii) f is increasing, strictly concave on R^+, and twice continuously differentiable on R^{++}; $f'(k) > 0$ and $f''(k) < 0$; and (iii) $\lim_{k\to 0} f'(0) = \infty$ and $\lim_{k\to+\infty} f'(0) = 0$. Let δ_k denote the fixed rate of capital depreciation. Markets are competitive; thus labor and capital earn their marginal products, and firms earn zero profits. The real interest rate, $r(t)$, and the real wage of labor, $w(t)$, are given as before by

$$r(t) + \delta_k = f'(k(t)), \quad w(t) = f(k(t)) - k(t)f'(k(t)). \tag{2.2.1}$$

For simplicity of expression, we normalize $N = 1$. There is some initial capital stock k_0 that is owned equally by all individuals at the initial time.

We assume that agents have perfect foresight with respect to all future events and capital markets operate frictionless. This assumption can be replaced with other ways, such as exact expectations, or adaptive expectations, of determining expected prices.[5] The government levies no taxes. Nominal prices – prices quoted in money terms – are perfectly flexible. This implies that they adjust immediately to clear product, factor and asset markets. Money is introduced by assuming that a central bank distributes at no cost to the population a per capita amount of fiat money $M(t) > 0$ in order to finance all government expenditures via seigniorage.[6] The scheme according to which the money stock evolves over time is deterministic and known to all agents. With μ being the constant net growth rate of the money stock, $M(t)$, evolves over time according:

$$\dot{M}(t) = \mu M(t), \quad \mu > 0.$$

The government expenditure in real terms per capita, $g(t)$, is given by

$$g(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t), \tag{2.2.2}$$

where $P(t)$ is the price of the output good and $m(t)$ stand for the real value of money per capita measured in units of the output good.

In this model, money acts as a pure store of value. The demand for money relies exclusively on the speculative conjecture that the future exchange value of money in terms of goods will be positive because people will express a positive demand for money in the future. In the absence of uncertainty and

with money not being explicitly required for transaction purposes, this "bubbly view" implies that in a monetary equilibrium the return on money needs to be equal to the return on competing assets such as claims on productive investment projects. At time t, each consumer receiving the per capita nominal money stock M believes that money will be exchanged at the expected future price. $p^m(t+\Delta t) > 0$, at time $t+\Delta t$. The price of money $p^m(t)$ is in terms of goods or it expresses the amount of goods that can be purchased by one unity of money in each period t. According to the definitions, we have: $P(t) = 1/p^m(t)$. A necessary condition for a monetary economy to exist is that the price of money must be positive. If money has a positive value, people have an opportunity to save in money. According to the definition of the price of money, the deflation rate $P(t)/P(t+\Delta t)$ coincides with the real return on money. In a competitive economy, the absence of arbitrage opportunities entails equality of return on assets

$$\frac{P(t)}{P(t+\Delta t)} - 1 = \left[f'(k(t+\Delta t)) - \delta_k\right]\Delta t.$$

When Δt approaches infinite small, we have

$$-\frac{\dot{P}(t)}{P(t)} = f'(k(t)) - \delta_k.$$

From $\dot{M}/P = \mu M/P = \mu m$, we have

$$\pi(t) \equiv \frac{\dot{P}(t)}{P(t)} = \mu - \frac{\dot{m}(t)}{m(t)}.$$

From the above two equations, we have

$$\frac{\dot{m}(t)}{m(t)} = f'(k(t)) + \mu - \delta_k. \tag{2.2.3}$$

It should be noted that equation (2.2.3) is equivalent to

$$f'(k(t)) - \delta_k + \pi(t) = 0.$$

This equation says that expected money yield on the capital good is equal to zero.[7] As we assume that money is held only for asset purposes and expected inflation rate is equal to the actual inflation rate, we see that the expected money yield on capital, $f' - \delta_k + \pi$, equals zero, the money yield on money. It should be noted that if $f' - \delta_k + \pi > 0$, and if the variance of expected returns does not matter, then everyone would want to hold his wealth in the form of capital, whereas if $f' - \delta_k + \pi < 0$, desired portfolios would consist only of money.

The consumer obtains real income in period t from the interest payment $r(t)k(t)$ and the wage payment $w(t)$

$$y(t) = r(t)k(t) + w(t).$$

We call $y(t)$ the current income. The sum of money that consumers are using for consuming, saving, or transferring are not necessarily equal to the current income because consumers can sell wealth to pay, for instance, the current consumption if the temporary income is not sufficient for buying goods. A retired person may live not only on the interest payment but also have to spend some of his wealth. The total real value of wealth and money, $a(t)$, that consumers can use to purchase goods and services and to save is equal to $k(t) + m(t)$. The cost of holding money due to the inflation is given by $\pi(t)m(t)$. We define the disposable income as the sum of $k(t) + m(t)$, the current income, $y(t)$, and the cost of holding money, $\pi(t)m(t)$, i.e.

$$\hat{y}(t) = y(t) + a(t) - \pi(t)m(t). \tag{2.2.4}$$

The disposable income is divided between the savings and consumption. We denote consumption and savings at t by $c(t)$ and $s(t)$, respectively. The budget constraint is given by

$$c(t) + s(t) = \hat{y}(t).$$

We use the utility function as in Section 1.2 to solve allocation of consumption and savings. We assume that utility level $U(t)$ that the consumers obtain is dependent on the consumption level $c(t)$ of commodity and the savings $s(t)$ as follows

$$U(t) = U(c(t), s(t)).$$

The consumer is to choose his most preferred bundle $(c(t), s(t))$ of consumption and saving under his budget constraint. The utility maximizing problem at any time is defined by

$$\underset{c(t),s(t)\geq 0}{Max}\ U(c(t), s(t))$$
$$\text{s.t. } c(t) + s(t) \leq \hat{y}(t). \tag{2.2.5}$$

As in Section 2.2, under certain conditions the problem has a global solution. We denote an optimal solution as function of the disposable income

$$(c(t), s(t)) = (c(\hat{y}(t)), s(\hat{y}(t))).$$

The vector, $(c(\hat{y}(t)), s(\hat{y}(t)))$ is known as the Walrasian (or ordinary or market) demand function, when it is single-valued for all positive disposable income.

We require U, a C^2 function, and satisfy $U_i > 0, U_{ii} < 0$, and $U_{cs} \geq 0$ for any $(c, s) > 0$. We know that $0 < ds/d\hat{y} < 1$ and $0 < dc/d\hat{y} < 1$ in the case of $U_{sc} \geq 0$ under the second-order condition of maximization.

The change in real wealth is equal to real savings minus real dissavings, that is

$$\dot{a}(t) = s(t) - a(t). \tag{2.2.6}$$

We have thus built the model. We now examine behavior of the model.

2.2.2 The dynamics and stead state

We now show that the motion of the system may be expressed in two-dimensional difference equations. Government spending is given by equation (2.2.2). From equation (2.2.4) and the definition of $a(t)$, we obtain

$$\hat{y}(t) = f(k(t)) + \delta k(t) + (1 - \mu)m(t) + \dot{m}(t),$$

where we use $y = f(k) - \delta_k k, \pi = \mu - \dot{m}/m$ and $\delta \equiv 1 - \delta_k$. From equation (2.2.6), $s = s(\hat{y})$, and the definition of $a(t)$, we obtain

$$\dot{k}(t) = s(\hat{y}(t)) - k(t) - m(t) - \dot{m}(t). \tag{2.2.7}$$

From equation (2.2.3), we obtain

$$\dot{m}(t) = \left[f'(k(t)) + \mu - \delta_k\right] m(t). \tag{2.2.8}$$

As $\dot{m}$ is a function of k and m, we see that $\hat{y}$ is also a function of k and m. Hence, the two differential equations, (2.2.7) and (2.2.8), contain two variables, k and m.

Definition 2.2.1
Given the initial capital stock k_0 and the initial money stock $M(0)$, a competitive equilibrium is given by a sequence of quantities $\{m(t), c(t), s(t), g(t), k(t)\}$ and a sequence of prices $\{r(t), w(t), P(t)\}$ such that for any t: (i) competition ensures that factors get paid their marginal products according to (2.2.1); (ii) given $\dot{M}(t) = \mu M(t)$, the budget constraint (2.2.2) of the government is satisfied; (iii) given the price consequence, agents solve optimally the decision problem; (iv) the evolution of money balances satisfy (2.2.8); (v) investments and savings are matched by (2.2.7); and (vi) all markets clear with the equilibrium conditions being as following:

labor market: $w(t) = f(k(t)) - k(t)f'(k(t))$;

money market: $m(t) = M(t)/P(t)$; and

goods market: $f(k(t)) + k(t) = s(t) - m(t) + \delta_k k(t) + c(t)$.

Equilibrium can be further classified as inside and outside money equilibrium: an inside money equilibrium is associated with a situation in which outside balances are zero; an outside money equilibrium is associated with positive outside money balances. Inside money equilibrium is the situation where the paper money has no value, which occurs for $P(t) = +\infty$.

Inside money equilibrium

We now examine dynamic properties of the difference equations, (2.2.7) and (2.2.8). At inside money equilibrium with $m(t) = 0$, we have

$$\dot{k}(t) = s(\hat{y}(t)) - k(t),$$

where

$$\hat{y}(t) = f(k(t)) + \delta k(t).$$

Government spending is reduced to zero. Hence, the model is identical to the model in Section 1.2 without money. In a stationary state

$$s(f(k) + \delta k) = k. \tag{2.2.9}$$

The following proposition is a restatement of Theorem 1.2.2.

Proposition 2.2.1
Assume the utility function satisfies Assumptions A.1 and A.2 in Section 2.2. Then the inside money economy has a unique stable equilibrium.

In the rest of this section, we denote the unique non-trivial steady state by $\bar{k}$.

Dynamic properties of the monetary economy

We are mainly interested in monetary economy which is characterized by the three difference equations (2.2.2), (2.2.7), and (2.2.8). As (2.2.7) and (2.2.8) do not contain $g(t)$, we treat $g(t)$ as "endogenous variable." In fact, once $m(t)$ is determined, we determine $g(t)$ by (2.2.2).[8] The dynamics of the monetary economy are described by (2.2.7) and (2.2.8) as follows

$$\begin{aligned} \dot{k}(t) &= s(\hat{y}(t)) - k(t) - m(t) - \dot{m}(t), \\ \dot{m}(t) &= [f'(k(t)) + \mu - \delta_k]m(t), \end{aligned} \tag{2.2.10}$$

where

$$\hat{y}(t) = f(k(t)) + \delta k(t) + (1 - \mu)m(t) + \dot{m}(t).$$

A stationary state of the monetary economy needs to satisfy

$$\begin{aligned} s(\hat{y}) - k - m &= 0, \\ f'(k) &= \delta_k - \mu. \end{aligned} \tag{2.2.11}$$

We assume $\delta_k > \mu$. From $f'(k) = \delta_k - \mu$, we determine a unique k, denoted by k^*. From the first equation in (2.2.11), we have

$$m = s\left[f(k^*) + \delta k^* + (1-\mu)m\right] - k^*. \tag{2.2.12}$$

To find solutions of (2.2.12), define

$$Z(m, k^*) \equiv \left[f(k^*) + \delta k^* + (1-\mu)m\right] - k^* - m.$$

As $Z(0, k^*) = s(f(k^*) + \delta k^*) - k^*$ and $\partial Z/\partial m < 0$, we see that if

$$Z(0, k^*) = s(f(k^*) + \delta k^*) - k^* \leq 0,$$

$Z(m, k^*) = 0$ has no positive solution. The equation $Z(m, k^*) = 0$ has a positive solution $m > 0$ if $Z(0, k^*) > 0$. If $k^* < \bar{k}$, then $Z\left(0, \bar{k}\right) = 0$ and

$$\frac{\partial Z}{\partial k} = (f'(k) + \delta)s' - 1 < 0, \quad k^* \leq k \leq \bar{k},$$

where we use $0 < s' < 1, f'(k^*) + \delta = 1 - \mu < 1$ and f' decreases in k.We thus have $Z(0, k^*) > 0$, if $k^* < \bar{k}$. We now have the following proposition.

Proposition 2.2.2
Suppose that the inside competitive equilibrium has a unique non-trivial steady state $\bar{k} > 0$. Then, the monetary economy has a unique steady state if the following inequality holds: $k^* < \bar{k}$. Moreover, the steady state is a saddle point.

What we need to confirm is that the steady state is a saddle point. The two eigenvalues, ϕ_1 and ϕ_2, are determined by

$$\phi^2 - \phi_0\phi + f''m = 0,$$

where

$$\phi_0 \equiv s'(f' + \delta) - f''m.$$

Accordingly, we have $\phi_{1,2} = \phi_0/2 \pm \sqrt{\phi_0^2/4 - f''m}$. As $f''m < 0$, one eigenvalue is positive and the other one is negative. We conclude that the unique equilibrium point is a saddle point.

The behavior of the monetary economy is illustrated in Fig. 2.2.1. In the k–m plane, the equation $f'(k) = \delta_k - \mu$ is a vertical line and it passes the curve $k = s[f(k) + \delta k + (1-\mu)m] - m$. The latter has a unique maximum. It is concave and passes through the origin and $(\bar{k}, 0)$. The unique monetary state is at the intersection of the two curves.

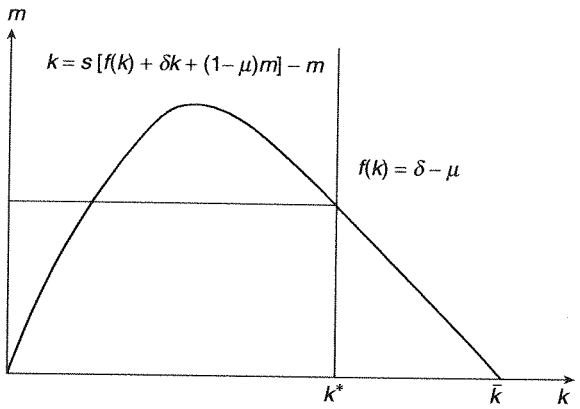


Figure 2.2.1 Steady state of monetary economy with capital accumulation.

2.2.3 Comparative statics in monetary economy

This section examines effects of changes in some parameters on the economic equilibrium.

Effects of inflation

First, we examine effects of changes in the monetary parameter, μ. From (2.2.11), we get

$$\frac{dk^*}{d\mu} = \frac{-1}{f''} > 0,$$

$$\frac{dm^*}{d\mu} = \left[\frac{(f'+\delta)s' - 1}{f''} + m^* s'\right] \frac{1}{(1-\mu)s' - 1} < 0,$$

where we use $(f'+\delta)s' - 1 < 0$ and $(1-\mu)s' - 1 < 0$,

If the rate of monetary expansion μ is permanently raised, then the per capita capital stock in the new monetary steady state is higher than before. That is, the Tobin effect prevails.[9] This is illustrated in Figure 2.2.2. As μ is shifted, the accumulation equation $s(\hat{y}) - k - m = 0$ is lowered for given level of m, except at the two extreme points which are not affected by μ; the arbitrage relation $f'(k) = \delta_k - \mu$ is shifted by lowering the return on real balances via increased inflation taxation. To rebalance the arbitrage relation, the composition of the portfolios of agents needs to readjust in favor of capital. In the new steady state, the capital stock will be higher than before. Acting like a tax on money, higher inflation reduces the return on real balances, and since assets are specified as substitutes, the portfolio effect ensures that in the new equilibrium a larger fraction

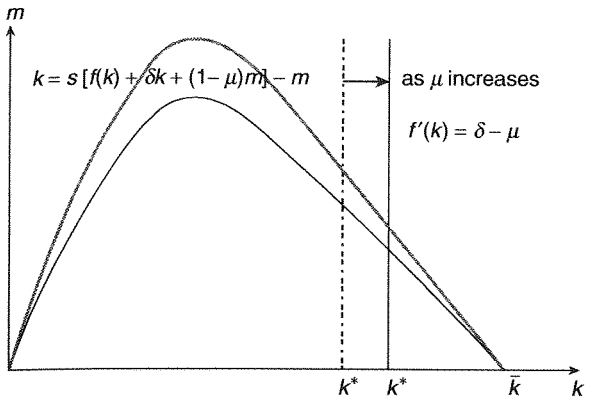


Figure 2.2.2 The Tobin effect.

of the portfolio will be invested in capital than before and the money holding is reduced.

From $a = s$, the impact on the total real wealth is

$$\frac{da}{d\mu} = \left[(f' + \delta) \frac{dk^*}{d\mu} + (1-\mu) \frac{dm^*}{d\mu} - m^* \right] s' = \left[(1-\mu) \frac{da}{d\mu} - m^* \right] s'.$$

Hence, we have

$$\frac{da}{d\mu} = -\frac{m^* s'}{1-(1-\mu)s'} < 0.$$

We conclude that an increase in μ reduces the total real wealth. We see that the government's inflationary policy will reduce the total real wealth. As

$$\frac{dc}{d\mu} = \frac{dc}{d\hat{y}} \frac{d\hat{y}}{d\mu} = \frac{dc}{d\hat{y}} \left[(1-\mu) \frac{da}{d\mu} - a \right] < 0,$$

the impact on the consumption level is negative.

Effects of technological change

To examine effects of technological change, we specify $f = Ak^{\alpha}$, where $0 < \alpha < 1$. From equations (2.2.11) we have

$$k^{\beta} = \frac{\alpha A}{\delta_k - \mu}.$$

From this equation and $k = s[f(k) + \delta k + (1-\mu)m] - m$, we have

$$\frac{dk}{dA} = -\frac{f'}{Af''} > 0,$$

$$\frac{dm}{dA} = \left(\frac{1-(f'+\delta)s'}{(1-\mu)s'-1}\right)\frac{dk}{dA} < 0,$$

where we use $1-(f'+\delta)s' > 0$ and $(1-\mu)s'-1 < 0$. We see that an improvement in the total productivity increases per capita capital intensity but reduces real money balances. We also have

$$\frac{dr}{dA} = \frac{f'}{A} + f''\frac{dk}{dA} = 0,$$

$$\frac{dw}{dA} = \beta f'\frac{dk}{dA} > 0.$$

The rate of interest is not affected and the wage rate is increased. The effects on the total wealth and consumption are given

$$\frac{da}{dA} = 0,$$

$$\frac{dc}{dA} = \frac{dc}{d\hat{y}}\frac{d\hat{y}}{dA} = 0.$$

An increase in the total productivity has no impact on the total wealth and per capita consumption. It should be noted that this result is independent of the production specification. For instance, if we have a production function, $f = f(k, A)$, where A is a measurement of productivity. It can be shown that if $f_k > 0$, $f_{kk} < 0$, and $f_{kA} > 0$, then all the above results on the effects of change in A hold. We see that in an economy in which money is held only for the store of value, then an improvement in productivity has no impact on the consumption and total wealth in the long term.[10]

We now explain how the total wealth is not affected. From the arbitrage relation $f'(k) = \delta_k - \mu$, we see that as the productivity is improved, the capital intensity should be increased. In fact, as $r = -\mu$ at steady state, we see that the real rate of interest at equilibrium is only affected by μ. With the other conditions fixed, an increase in the productivity increases the real return of capital and an increase in the capital intensity tends to reduce the return. Hence to maintain the system at the new equilibrium the capital intensity falls. We note that at steady state we have

$$\hat{y} = f + \delta k + (1-\mu)m = f - \delta_k k + \mu k + (1-\mu)a.$$

Hence, with $f' - \delta_k + \mu = 0$ we have

$$\frac{d\hat{y}}{dA} = (f' - \delta_k + \mu)\frac{dk}{dA} + (1-\mu)\frac{da}{dA} = (1-\mu)\frac{da}{dA}.$$

We see that the change in the total wealth is slower than the change in the disposable income. Taking derivatives of the equilibrium condition $s(\hat{y}) = a$ yields

$$\left[(1-\mu)\frac{ds}{d\hat{y}} - 1\right]\frac{da}{dA} = 0.$$

At steady state we have $da/dA = 0$. As the capital intensity rises, per capita money holdings falls.

Effects of the propensity to save

To examine effects of the propensity to save, we specify $U = c^{\xi_0} s^{\lambda_0}$, where $\xi_0, \lambda_0 > 0$. In this case, we have $s = \lambda\hat{y}$, where $\lambda \equiv \lambda_0/(\xi_0 + \lambda_0)$. From $f'(k) = \delta_k - \mu$ and $k = \lambda[f(k) + \delta k + (1-\mu)m] - m$, we have

$$\frac{dk}{d\lambda_0} = 0,$$

$$\frac{dm}{d\lambda_0} = \frac{\xi_0}{(\xi_0+\lambda_0)^2}\left(\frac{f+\delta k+(1-\mu)m}{1-(1-\mu)\lambda}\right) > 0.$$

As the propensity to save rises, the per capita capital intensity is not affected and the per capita real money balances is increased. The rate of interest and wage rate are not affected by the change. The effects on the total wealth and consumption are given

$$\frac{da}{d\lambda_0} = \frac{dm}{d\lambda_0} > 0,$$

$$\frac{dc}{d\lambda_0} = \frac{dc}{d\hat{y}}\frac{d\hat{y}}{d\lambda_0} = (1-\mu)\frac{dc}{d\hat{y}}\frac{dm}{d\lambda_0} > 0.$$

2.3 A portfolio balance growth model

As in the Tobin model, the model in Section 2.2 considers that the economy has two assets. This section considers that the economy has three assets, capital, money, and bonds. The early growth models with the three assets are developed by, for instance, Stein (1966) and Villanueva (1971).[11]

2.3.1 The model

This section extends the model in Section 2.2 to include government bonds. Now there are three stores of value in which savings may be held: money, capital, and bonds. Money and bonds are government debt. Changes in their quantities

are transmitted to the private sector through government transfers. The variables, $N(=1), K(t), F(K(t), N), y(t) = f(k(t)), k(t), \mu, \delta_k, M(t)$, and $m(t)$ are defined as in Section 2.2. As before, we have

$$r(t) + \delta_k = f'(k(t)), w(t) = f(k(t)) - k(t)f'(k(t)), \tag{2.3.1}$$
$$\dot{M}(t) = \mu M(t), \quad \mu > 0,$$

$$g(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t), \tag{2.3.2}$$
$$\pi(t) \equiv \frac{\dot{P}(t)}{P(t)} = \mu - \frac{\dot{m}(t)}{m(t)},$$
$$\frac{\dot{m}(t)}{m(t)} = f'(k(t)) + \mu - \delta_k. \tag{2.3.3}$$

We introduce the government bonds, denoted by $B(t)$. In the economic literature, there are usually two polar assumptions. One is that B are infinitesimally short bonds, with a fixed nominal price, for instance, $P_b = 1$. The other is that B are perpetuities, paying a fixed coupon (unity, say) per unit of time, with the price being inversely related to the nominal interest rate, $P_b = 1/r$. If rational expectations and risk neutrality are assumed, then the nominal interest rate and the price of a bond with a unit coupon has the following relation:

$$r(t) = \frac{1}{P_b(t)} + \frac{\dot{P}_b(t)}{P_b(t)}, \quad t \leq T, \quad P_b(T) = \overline{P}_b,$$

where T is the time of maturity and $\overline{P}_b$ is the par value. This equation means that prior to the maturity date, risk-neutral arbitrage ensures that the short-term interest rate equals the rate of return earned on the coupon, $1/P_b$, plus the rate of capital gain, $\dot{P}_b/P_b$. At its maturity date, the bond is redeemed at its face value, $\overline{P}_b$.[12]

Assume for simplicity $P_b = 1$. The government budget constraint is expressed in nominal terms by

$$\dot{M}(t) + \dot{B}(t) = P(t)(g(t) - \tilde{\tau}(t)) + r(t)B(t), \tag{2.3.4}$$

in which $g(t)$ and $\tilde{\tau}(t)$ are real government expenditure and real taxes. The right-hand side is the government deficit in nominal terms. This deficit is financed by issuing additional money and/or selling more bonds. We can rewrite the government budget constraint as follows

$$\dot{m} + \dot{b} = g - \tilde{\tau} + rb - \pi(m + b), \tag{2.3.5}$$

where we use $m(t) \equiv M(t)/P(t)$ and $b(t) \equiv B(t)/P(t)$, and

$$\frac{\dot{M}}{P} = \dot{m} + \pi m, \frac{\dot{B}}{P} = \dot{b} + \pi b, \tag{2.3.6}$$

in which $\pi(t) \equiv \dot{P}(t)/P(t)$. From this equation, we see that $-\pi(t)(m(t) + b(t))$ serves as a source of real revenue to the government. The term is the inflation tax on the forms of government debt that are denominated in nominal terms.

The government faces with its budget constraint. The government has four policy instruments – debt, money, taxes, and government expenditure, and three instruments can be chosen independently. To close the model, the government policy has to be specified. If the government determines the fiscal instruments, g and $\tilde{\tau}$, exogenously, the government decides the mix between bond financing and money financing of the deficit. As discussed in Turnovsky (2000: Chapter 2), different policy instruments are chosen in different economic models. These instruments include: (1) holding the real stock of money constant, that is, $m(t) = \overline{m}$, which can also be expressed as $M(t) = \overline{m}P(t)$; (2) allowing the nominal stock of money to grow at a fixed rate, that is, $\dot{M}(t) = \mu M(t)$, which may also be expressed as $\dot{m}(t) = (\mu - \pi(t))m(t)$; and (3) keeping the real stock of government bonds constant, that is, $b(t) = \overline{b}$, which can also be represented in nominal terms, $B(t) = \overline{b}P(t)$. This section is concerned with the policy of $\dot{M}(t) = \mu M(t)$, which is already specified in equation (2.3.2).[13]

We now describe the household behavior. The current income, y_m, in nominal terms is expressed in the following form

$$y_m = rPk + Pw + iB - \pi B - \pi M - P\tilde{\tau}, \tag{2.3.7}$$

where i is the nominal interest rate.[14] The disposable income, $\hat{y}_m$, in nominal terms is

$$\hat{y}_m = y_m + a_m, \tag{2.3.8}$$

where a_m is nominal wealth given by

$$a_m \equiv Pk + B + M.$$

The disposable income in real terms, $\hat{y}(t)$, is given by $\hat{y} = \hat{y}_m/P$ and the real wealth, $a(t)$, is given by $a \equiv a_m/P$.

We assume that the utility level, $U(t)$, that the consumers obtain is dependent on the consumption level of commodity, $c(t)$, and the saving, $s(t)$: $U(t) = U(c(t), s(t))$. A typical consumer is to choose his most preferred bundle $(c(t), s(t))$ of consumption and saving under his budget constraint. The utility-maximizing problem at any time is defined by

$$\underset{c,s \geq 0}{\text{Max}}\, U(c(t), s(t))$$

$$\text{s.t. } c(t) + s(t) = \hat{y}(t).$$

We denote an optimal solution as function of the disposable income

$$(c, s) = (c(\hat{y}), s(\hat{y})).$$

We see that $0 < ds/d\hat{y} < 1$ and $0 < dc/d\hat{y} < 1$ in the case of $U_{sc} \geq 0$ under the second-order condition of maximization.

The evolution of $a(t)$ is given by

$$\dot{a}(t) = s(t) - a(t). \tag{2.3.9}$$

It should be noted that from equation (2.3.9), $s = \hat{y} - c$ and the definitions of $\hat{y}$, we have

$$c + \dot{k} + \dot{b} + \dot{m} = rk + w + rb - \pi m - \tilde{\tau},$$

in which we use $r = i - \pi$. This equation is identical to the budget constraint in the Sidrauski model in the representative agent approach presented in Turnovsky (2000: 265). We have thus built the model.

2.3.2 The dynamics and equilibrium of the monetary economy

First, we need to find equations, which describe motion of the dynamic system. From equation (2.3.3), we have

$$\dot{m} = \left[f'(k) + \mu - \delta_k\right] m. \tag{2.3.10}$$

Substituting the definition of $a(t)$ and

$$\hat{y} = (1 + r)k + w + (1 + r)b + (1 - \mu)m + \dot{m} - \tilde{\tau},$$

into equation (2.3.9) yields

$$\dot{k} = s(\hat{y}) - k - m - b - \dot{m} - \dot{b}. \tag{2.3.11}$$

As $\dot{m}(t)$ is a function of k and m, and r and w are functions of $k(t)$, we see that $\dot{k}$ is a function of $b, \dot{b}, k$, and m. From equation (2.3.5) and $\pi = \mu - \dot{m}/m$, we have

$$\dot{b} = g - \tilde{\tau} + rb - \left(\mu - \frac{\dot{m}}{m}\right)(m + b) - \dot{m}. \tag{2.3.12}$$

The dynamics consist of three differential equations, (2.3.10)–(2.3.12), with three endogenous variables, $k(t)$, $m(t)$, and $b(t)$, and two time-dependent parameters, $g(t)$ and $\tilde{\tau}(t)$. For simplicity of analysis, we assume that g and $\tilde{\tau}$ are constant over time.

We now examine dynamic properties of the differential equations. We will not be concerned with inside money equilibrium with $m(t) = 0$. From equations (2.3.10)–(2.3.12), a steady state is determined by

$$\begin{aligned} &f'(k) = \delta_k - \mu, \\ &s(\hat{y}) = k + m + b, \\ &\tau_g + rb - \mu(m + b) = 0, \end{aligned} \tag{2.3.13}$$

where $\tau_g \equiv g - \tilde{\tau}$ and

$$\hat{y} = (1+r)k + w + (1+r)b + (1-\mu)m - \tilde{\tau}.$$

We assume that the government expenditure is always larger than the tax, i.e. $\tau_g > 0$. As in Section 2.2, we assume $\delta_k > \mu$. The equation, $f'(k) = \delta_k - \mu$, uniquely determines k. From $r = -\mu$ and the last equation in (2.3.13), we solve b as a function of m as follows

$$b = \frac{\tau_g - \mu m}{2\mu}. \tag{2.3.14}$$

It is straightforward to check

$$\hat{y} = k_0 + \mu_0 m \tag{2.3.15}$$

where $\mu_0 \equiv (1-\mu)/2$ and

$$k_0 \equiv 2\mu_0 k + w + \frac{\mu_0 \tau_g}{\mu} - \tilde{\tau}.$$

Substituting equations (2.3.14) and (2.3.15) into the second equation in (2.3.13) yields

$$\Omega(m) \equiv s(k_0 + \mu_0 m) - \frac{m}{2} - k - \frac{\tau_g}{2\mu} = 0. \tag{2.3.16}$$

The number of steady states of the monetary economy is the same as the number of solutions of $\Omega(m) = 0$, for $m > 0$. First, we notice

$$\Omega'(m) = \left(\frac{1-\mu}{2}\right) s' - \frac{1}{2} < 0,$$

and $\Omega(m)$ should be negative for sufficiently large m.[15] Hence, if $\Omega(0) > 0$, then the equation has a unique solution. This condition is guaranteed if

$$\Omega(0) = s\left(2\mu_0 k + w + \frac{\mu_0 \tau_g}{\mu} - \tilde{\tau}\right) - k - \frac{\tau_g}{2\mu} > 0. \tag{2.3.17}$$

It can be seen that we can discuss the existence of solution to equation (2.3.17) as the model in Section 2.2. The dynamic system has a unique steady state. The eigenvalues, ϕ, are given by

$$\begin{vmatrix} -\phi & f''m & 0 \\ k_1 & k_2-\phi & k_1+\mu \\ -\mu & k_b & -2\mu-\phi \end{vmatrix} = 0$$

In which we use $r=-\mu$ and

$$k_b \equiv 2f''b-\mu(m+b)<0, \quad k_1 \equiv (1-\mu)(s'-1)<0,$$
$$k_2 \equiv (1-\mu+f''b+f''m)s'-1-f''m-k_b,$$

where we require $0<\mu<1$. Hence, the eigenvalues are solutions to the following equation

$$\phi^3+(2\mu-k_2)\phi^2-a_1\phi+a_2=0,$$

where

$$a_1 \equiv k_1k_b+\mu k_b+k_1f''m+2\mu k_2,\ a_2 \equiv (\mu-k_1)\mu f''m<0.$$

By the definitions of k_1, k_2, and k_b, we have

$$a_1=(k_b+2\mu)k_1-[(1-s')2f''b-\mu(m+b)]\mu$$
$$+(2-\mu)(s'-1)f''m-2\mu^2.$$

As $k_b, k_1, f'', 1-s'<0$, we see that if μ is properly small and $k_b+2\mu\le 0$, then we should have $a_1>0$. In the remainder of this section, we require $a_1>0$. For three eigenvalues, ϕ_j, we have

$$\phi_1\phi_2+\phi_2\phi_3+\phi_1\phi_3=-a_1<0, \quad \phi_1\phi_2\phi_3=-a_2>0.$$

The inequality, $\phi_1\phi_2+\phi_2\phi_3+\phi_1\phi_3<0$, implies that at least one eigenvalue's real part is negative. The inequality, $\phi_1\phi_2\phi_3>0$, implies that at least one eigenvalue is positive. These results imply that the unique steady point is a saddle point.

Proposition 2.3.1
The unique steady state is a saddle point.

2.3.3 Comparative statics in monetary economy

This section effects of changes in some parameters on the steady state which is determined by

$$\begin{aligned} & f'(k) = \delta_k - \mu, \\ & s[(1-\mu)a + w - \tilde{\tau}] = a, \\ & \frac{\tau_g}{\mu} - (a - k + b) = 0, \end{aligned} \tag{2.3.18}$$

where $\hat{y} = (1-\mu)a + w - \tilde{\tau}$ and $a = k + m + b$. The three equations contain k, a, and b, as variables. We solve the real money holding by $a = k + m + b$.

Effects of inflation

First, we examine effects of changes in the monetary parameter, μ. From (2.3.18), we get

$$\begin{aligned} \frac{dk}{d\mu} &= \frac{-1}{f''} > 0, \\ \frac{da}{d\mu} &= -\left[\frac{a-k}{1-(1-\mu)s'}\right]s' < 0, \\ \frac{db}{d\mu} &= -\frac{\tau_g}{\mu^2} + \left[\frac{a-k}{1-(1-\mu)s'}\right]s' + \frac{dk}{d\mu}, \\ \frac{dm}{d\mu} &= \frac{\tau_g}{\mu^2} - \left[\frac{a-k}{1-(1-\mu)s'}\right]2s' - 2\frac{dk}{d\mu}, \end{aligned}$$

where we use $(1-\mu)s' - 1 < 0$. If the rate of monetary expansion μ is permanently raised, then the per capita capital stock in the new monetary steady state is higher than before. That is, the Tobin effect prevails. The total wealth is reduced. The effects of real money holdings and bonds are ambiguous. As

$$\frac{dc}{d\mu} = \frac{dc}{d\hat{y}}\left((1-\mu)\frac{da}{d\mu} - a + k\right) < 0,$$

the impact on the consumption level is negative. The effects on the real rate of interest and wage rate are

$$\begin{aligned} \frac{dr}{d\mu} &= -1 < 0, \\ \frac{dw}{d\mu} &= \beta f' \frac{dk}{d\mu} > 0. \end{aligned}$$

The rate of interest falls and the wage rate rises.

Effects of technological change

To examine effects of technological change, we specify $f = Ak^{\alpha}$, where $0 < \alpha < 1$. From equations (2.3.18), we have

$$\frac{dk}{dA} = -\frac{f'}{Af''} > 0.$$

$$\frac{da}{dA} = \frac{s'f}{\left(1 - s'(1-\mu)\right)A} > 0,$$

$$\frac{db}{dA} = \left[1 - (1-\alpha)\alpha s' - (1-\mu)s'\right]\frac{k}{(1-\alpha)A},$$

$$\frac{dm}{dA} = -2\frac{db}{dA}.$$

We see that an improvement in the total productivity increases per capita capital intensity and the total wealth. The sign of change in real bonds holdings is the same as that of $1 - (1-\alpha)\alpha s' - (1-\mu)s'$. If the marginal save, s', is high, then real bonds holding tends to be positive. The change in real money holdings is the opposite to the sign of db/dA. We also have

$$\frac{dr}{dA} = \frac{f'}{A} + f''\frac{dk}{dA} = 0,$$

$$\frac{dw}{dA} = \beta f'\frac{dk}{dA} > 0.$$

The rate of interest is not affected and the wage rate is increased. The effects on the total wealth and consumption are given

$$\frac{dc}{dA} = \left[(1-\mu)\frac{da}{dA} + \beta f'\frac{dk}{dA}\right]\frac{dc}{d\hat{y}} > 0.$$

An increase in the total productivity increases the per capita consumption.

Effects of the propensity to save

To examine effects of the propensity to save, we specify $U = c^{\xi_0}s^{\lambda_0}$, where $\xi_0, \lambda_0 > 0$. In this case, we have $s = \lambda\hat{y}$, where $\lambda \equiv \lambda_0/(\xi_0 + \lambda_0)$. From equations (2.3.18), we have

$$\frac{dk}{d\lambda_0} = 0,$$

$$\frac{da}{d\lambda_0} = \frac{1}{2}\frac{dm}{d\lambda_0} = -\frac{db}{d\lambda_0} = \frac{a\xi}{[1-(1-\mu)\lambda]\lambda_0} > 0.$$

A rise in the propensity to save has no impact on per capita physical wealth, rises the total real wealth and real money holdings, but reduces real bonds. The change

has no impact on the real rate of interest and the wage rate. The effect on the consumption is given by

$$\frac{dc}{d\lambda_0} = (1-\mu)\xi\frac{da}{d\lambda_0} - \frac{c}{(\xi_0+\lambda_0)}.$$

The impact on per capita consumption is ambiguous.

Effects of the government expenditure and tax

The effects of changes in the government expenditures are given by

$$\frac{dk}{dg} = \frac{da}{dg} = 0,$$
$$\frac{db}{dg} = -\frac{dm}{dg} = \frac{1}{\mu}.$$

The total wealth and per capita physical wealth are not affected by changes in the government expenditure. As the government expenditures are increased, real bonds rise and real money holdings falls. It is direct to check that k, r, w, and c are not affected.

The effects of changes in tax are given by

$$\frac{dk}{d\tilde{\tau}} = 0,$$
$$\frac{da}{d\tilde{\tau}} = -\frac{s'}{1-(1-\mu)s'} < 0,$$
$$\frac{db}{d\tilde{\tau}} = \frac{-1+s'}{[1-(1-\mu)s']\mu} < 0,$$
$$\frac{dm}{d\tilde{\tau}} = \frac{1-s'-\mu s'}{[1-(1-\mu)s']\mu}.$$

The per capita physical wealth is not affected and the total wealth and real bonds are reduced. If μ is small, the term, $\mu s'$, is small. If $1-s'-\mu s'$ is positive, then real money holdings rises. The real rate of interest and wage rate are not affected. The effect on the consumption is given by

$$\frac{dc}{d\tilde{\tau}} = \left[(1-\mu)\frac{da}{d\tilde{\tau}} - 1\right]\frac{dc}{d\hat{y}} < 0.$$

An increase in tax reduces per capita consumption.

2.4 Monetary growth with two types of households

Easterly and Fischer (2001: 164) point out the importance of studying the effects of inflation policy upon different people in the following way:

> The claim that "inflation is the cruelest tax of all" is often interpreted as meaning that inflation hurts the poor relatively more than the rich. It could also mean that the inflation tax is particularly unfair because, the taxing mechanism being little understood, the inflation tax can be imposed by stealth.

They also point out some concrete situations for difference of effects on the rich and the poor. For instance, as the rich and more sophisticated are likely to have better access to financial instruments that tend to hedge inflation and the small portfolios of the poor tend to have a larger share of cash, the poor may be hurt relatively far more than the rich by inflation. Among the elderly poor, pensions are often not fully indexed and so inflation will directly reduce their real incomes. In fact, the relative effects of inflation on different people are dependent on the institutions and stage of economic development. As many of the effects of inflation come through complicated details of the tax system and government distributional policy, analytical conclusions may well differ among economists.[16] The Ramsey model and its variations are the main framework for economists to analyze consumer decisions over time. Nevertheless, this modeling framework is not effective for analyzing economic issues related to monetary issues because even a simple economic problem tends to result in an analytically intractable dynamic model.

Irrespective of widely observed changes of inequalities in income and wealth in different societies, economics still lacks proper theories for examining dynamics of income and wealth distribution over time and space. Atkinson and Bourguignon (2000: 5) describe the current situation as follows:

> No unified theory of income distribution actually exists. Even though several titles of books and articles announce quite ambitiously the state of such a "theory of income distribution," they typically refer to only one part of what actually be covered by such a theory: the determination of wages in the labor market, factor shares, the accumulation of wealth, etc. Rather than a unified theory, the literature thus offers a series of building blocks with which distribution issues are to be studied. Because of the natural complexity of the subject, however, no serious attempt at integrating them has really been made.[17]

As shown in Zhang (2005a, 2006a), the OSG modeling framework enables us to make, at least, one serious attempt at integrating different ideas. This book will also apply the OSG framework to examine issues of income and wealth distribution among heterogeneous households in monetary economy.

2.4.1 The model of two types of households

Let there be two groups of consumers, indexed by j, in the economy. Each type of consumer has a fixed number of the population, denoted by N_j. The aggregated labor force N is given by

$$N = z_1 N_1 + z_2 N_2,$$

where z_j are the level of human capital of group j, $j = 1, 2$. We assume z_j to be fixed. We now study behavior of producers. Let $K(t)$ denote the capital stock at time t and N the flow of labor services used at t for production. We use the conventional production function to describe a relationship between inputs and output. The rate of interest and wage rate are determined by markets. Hence, for any individual firm r and w_j are given at each point of time. The production sector chooses the two variables, K and N, to maximize its profit. Denote $\delta_k (0 \leq \delta_k < 1)$ the depreciation rate of physical capital. The marginal conditions are given by

$$\begin{aligned} &r.(t) + \delta_k = f'(k(t)), \\ &w_j(t) = z_j w(t), \quad w(t) \equiv f(k(t)) - k(t) f'(k(t)). \end{aligned} \tag{2.4.1}$$

The ratio of wage rates is equal to the ratio of the human capital levels between the two groups.

The government levies no taxes. Money is introduced by assuming that a central bank distributes at no cost to the population a per capita amount of fiat money $M(t) > 0$. The scheme according to which the money stock evolves over time is deterministic and known to all agents. With μ being the constant net growth rate of the money stock, $M(t)$ evolves over time according to

$$\dot{M}(t) = \mu M(t), \quad \mu > 0.$$

The government expenditure in real terms per capita, $g(t)$, is given by

$$g(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t), \tag{2.4.2}$$

where $P(t)$ is the price of the output good and $m(t)$ stand for the real value of money per capita measured in units of the output good. Household j receives $\mu_j m(t)$ units of paper money from the government through a "helicopter drop" to each group, also considered independent of his money holdings.[18] The inflation rate, $\pi(t)$, is given by

$$\pi(t) \equiv \frac{\dot{P}(t)}{P(t)} = \mu - \frac{\dot{m}(t)}{m(t)}. \tag{2.4.3}$$

Equation (2.2.3) also holds in this model. We have

$$\frac{\dot{m}(t)}{m(t)} = f'(k(t)) + \mu - \delta_k. \tag{2.4.4}$$

Household j's disposable income in real terms, $\hat{y}_j(t)$, is

$$\hat{y}_j(t) = r(t)k_j(t) + w_j(t) + \mu_j m(t) - \pi(t)m_j(t) + a_j(t), \tag{2.4.5}$$

where $a_j(t) \equiv k_j(t) + m_j(t)$. The total money issued equals the total money received by the two groups

$$N_1\mu_1 m(t) + N_2\mu_2 m(t) = N_0\mu m(t).$$

where $N_0(\equiv N_1 + N_2)$ is the population. We require $\mu_j \geq 0$ and rewrite the above equation as

$$\overline{n}_1\mu_1 + \overline{n}_2\mu_2 = \mu,$$

where $\overline{n}_j \equiv N_j/N_0$. We assume that at each point of time consumers' preferences over money, consumption, and saving can be represented by the following utility function

$$U_j(t) = c_j^{\xi_{0j}}(t)s_j^{\lambda_{0j}}(t), \quad \xi_{0j}, \lambda_{0j} > 0,$$

where ξ_{0j} household j' propensity to consume, and λ_{0j} propensity to own wealth. The budget constraint is given by

$$c_j(t) + s_j(t) = \hat{y}_j(t).$$

Consumers' problem is to choose current money, consumption, and savings in such a way that utility levels are maximized. Maximizing $U_j(t)$ subject to the budget constraint yields

$$c_j = \xi_j\hat{y}_j, s_j = \lambda_j\hat{y}_j, \tag{2.4.6}$$

where

$$\xi_j \equiv \rho_j\xi_{0j}, \quad \lambda_j \equiv \rho_j\lambda_{0j}, \quad \rho_j \equiv \frac{1}{\xi_{0j} + \lambda_{0j}}.$$

The real wealth changes as follows

$$\dot{a}_j(t) = s_j(t) - a_j(t). \tag{2.4.7}$$

The total demand for money is equal to the total supply

$$N_0 m(t) = m_1(t)N_1 + m_2(t)N_2. \tag{2.4.8}$$

We assume that household j's nominal money, $M_j(t)$, is proportional to $M(t)$ with the proportional rate, μ_j/μ, as follows: $M_j(t) = \mu_j M(t)/\mu$.[19] That is

$$m_j(t) = \frac{\mu_j m(t)}{\mu}. \tag{2.4.9}$$

Under $\bar{n}_1\mu_1 + \bar{n}_2\mu_2 = \mu$, equation (2.4.9) implies equation (2.4.8). The assumption of full employment of capital is given by

$$K(t) = k_1(t)N_1 + k_2(t)N_2,$$

that is

$$k(t) = n_1 k_1(t) + n_2 k_2(t), \tag{2.4.10}$$

where $n_j \equiv N_j/N$. This equation expresses $k(t)$ as a unique function of $k_1(t)$ and $k_2(t)$. We have thus built the model.

2.4.2 The motion and steady state

From (2.4.4), we directly get the motion of m

$$\dot{m} = f_1(k)m, \tag{2.4.11}$$

where $f_1(k) \equiv f'(k) + \mu - \delta_k$. Substituting $a_j = k_j + m_j$ and $s_j = \lambda_j \hat{y}_j$ into equations (2.4.7)

$$\dot{k}_j = (\lambda_j r - \xi_j)k_j + \lambda_j w_j + \lambda_j \mu_j m - (\xi_j + \lambda_j \pi)m_j - \dot{m}_j, \tag{2.4.12}$$

where we use equations (2.4.5). Substitute equation (2.4.3) and $m_j = \mu_j m/\mu$ into equations (2.4.12)

$$\dot{k}_j = (\lambda_j r - \xi_j)k_j + \lambda_j w_j - (1 + f_1(k))\frac{\xi_j \mu_j m}{\mu}. \tag{2.4.13}$$

Lemma 2.4.1

The dynamics are given by the three differential equations, (2.4.11) and (2.4.13), with three variables, $m(t), k_1(t)$, and $k_2(t)$. For any meaningful solution of the dynamic system, all the variables are determined by the following procedure: k by (2.4.10) $\to K = kN \to f = f(k) \to \pi$ by (2.4.6) $\to r, w$, and w_j by (2.4.1) $\to m_j$ by (2.4.9) $\to \dot{P}/P = \pi \to M_j = \mu_j M/\mu \to a_j = k_j + m_j \to c_j$ and s_j by (2.4.6).

By equations (2.4.11) and (2.4.13) at steady state we have[20]

$$f'(k) + \mu - \delta_k = 0,$$

$$(\lambda_j r - \xi_j)k_j + \lambda_j w_j - \frac{\xi_j \mu_j m}{\mu} = 0. \tag{2.4.14}$$

From the first equation in (2.4.14), we uniquely determine k for $\delta_k > \mu$. For positive k, we uniquely determine r and w_j.

Multiplying the equation for household j in (2.4.14) by n_j and then adding the resulted two equations, we have

$$m = (k + \lambda_0)\xi_0, \tag{2.4.15}$$

where we use $k = n_1 k_1 + n_2 k_2$ and

$$\lambda_0 \equiv \sum_j \frac{n_j \lambda_j w_j}{\lambda_j r - \xi_j}, \quad \xi_0 \equiv \left(\sum_j \frac{n_j \xi_j \mu_j}{\mu (\lambda_j r - \xi_j)} \right)^{-1}.$$

From equation (2.4.15) and the last two equations in (2.4.14), we solve

$$k_j = \frac{\xi_j \mu_j m/\mu - \lambda_j w_j}{\lambda_j r - \xi_j}. \tag{2.4.16}$$

As $r = -\mu$, for m and k_j to be positive we should require

$$k + \lambda_0 < 0, \quad \frac{\xi_j \mu_j m}{\mu} < \lambda_j w_j.$$

It is straightforward to confirm that if

$$k > \lambda_{\min} \equiv \max\left[\left(\lambda_2 z_2 - \frac{\xi_2 \mu_2 \lambda_1 z_1}{\xi_1 \mu_1} \right) \frac{n_2 w}{\lambda_2 \mu + \xi_2}, \left(\lambda_1 z_1 - \frac{\xi_1 \mu_1 \lambda_2 z_2}{\xi_2 \mu_2} \right) \frac{n_1 w}{\lambda_1 \mu + \xi_1} \right], \tag{2.4.17}$$

then $\xi_j \mu_j m/\mu < \lambda_j w_j$. For instance, if we specify $f = Ak^\alpha$, $\mu_1 = \mu_2$, and $\lambda_1 = \lambda_2$, then the above inequality becomes

$$\frac{\alpha A}{\delta_k - \mu} > \frac{\beta \lambda}{\lambda \mu + \xi} Max\left[-(z_1 - z_2) n_2, -(z_2 - z_1) n_1 \right].$$

We see that the inequality is guaranteed if the difference between the two groups' human capital levels is small. As $k + \lambda_0 < 0$ is guaranteed if

$$k < \lambda_{\max} \equiv \sum_j \frac{n_j \lambda_j w_j}{\lambda_j \mu + \xi_j}.$$

Hence, if

$$\lambda_{\max} > k > \lambda_{\min},$$

the dynamic system has a unique steady state. The three eigenvalues, ϕ_j, are determined by

$$\phi^3 - A_1 \phi^2 + A_2 \phi + A_3 = 0,$$

where

$$A_1 \equiv b_4 + b_1, \quad A_2 \equiv \frac{(\xi_2\mu_2 n_2 + \xi_1\mu_1 n_1) f'' m}{\mu} - b_2 b_3 + b_1 b_4,$$

$$A_3 \equiv \left[(n_1 b_2 - n_2 b_1)\xi_2\mu_2 + (n_2 b_3 - n_1 b_4)\xi_1\mu_1\right]\frac{f'' m}{\mu},$$

$$b_1 \equiv \lambda_1 r - \xi_1 + \frac{n_1 b_2}{n_2}, \quad b_2 \equiv (k_1 - z_1 k)\lambda_1 n_2 f'' - \frac{n_2 f'' \xi_1 \mu_1 m}{\mu},$$

$$b_3 \equiv (k_2 - z_2 k)\lambda_2 n_1 f'' - \frac{n_1 f'' \xi_2 \mu_2 m}{\mu}, \quad b_4 \equiv \lambda_2 r - \xi_2 + \frac{n_2 b_3}{n_1}.$$

It is straightforward to check

$$A_1 = b_4 + b_1 = -(\lambda_1\mu + \xi_1 + \lambda_2\mu + \xi_2)$$
$$+ \left[k_1\lambda_1 n_1 + \lambda_2 n_2 k_2 - (z_1\lambda_1 n_1 + z_2\lambda_2 n_2)k\right]f'' - (n_1\xi_1\mu_1 + n_2\xi_2\mu_2)\frac{f'' m}{\mu},$$

$$A_3 \equiv -\left[(\lambda_1 r - \xi_1)n_2\xi_2\mu_2 + (\lambda_2 r - \xi_2)n_1\xi_1\mu_1\right]\frac{f'' m}{\mu} < 0.$$

It is not strict to require $A_1 < 0$. It is straightforward to check that if $\lambda_1 = \lambda_2$, then we have $A_1 < 0$. We also have $A_1 < 0$, if, for instance[21]

$$k_1\lambda_1 n_1 + \lambda_2 n_2 k_2 - (z_1\lambda_1 n_1 + z_2\lambda_2 n_2)k = \left(\frac{k_1}{z_1} - \frac{k_2}{z_2}\right)(\lambda_1 - \lambda_2)z_1 z_2 n_1 n_2 \geq 0.$$

This condition says that $A_1 < 0$ if $\lambda_1 > (<)\lambda_2$, then group 1's per human capital's physical wealth is higher (lower) than group 2's. From

$$\phi_1 + \phi_2 + \phi_1 = A_1 < 0, \phi_1\phi_2\phi_3 = -A_3 > 0.$$

The inequality, $\phi_1 + \phi_2 + \phi_1 = A_1 < 0$, implies that at least one eigenvalue's real part is negative. The inequality, $\phi_1\phi_2\phi_3 > 0$, implies that at least one eigenvalue is positive. These results imply that the unique steady point is a saddle point.

Proposition 2.4.1
If $\lambda_{max} > k > \lambda_{min}$, where k is uniquely given by $f'(k) + \mu - \delta_k = 0$, the dynamic monetary economic system has a unique steady state. The steady state is a saddle point if $A_1 < 0$.[22] The variables, m, k_1, and k_2, are given by (2.4.15) and (2.4.16). The rest of the variables are determined by the procedure in Lemma 2.4.1.

2.4.3 Comparative statics analysis

First, we examine effects of change in the inflation rate on the economy. From equations (2.4.1) and (2.4.14), we obtain

$$\frac{dk}{d\mu} = -\frac{1}{f''} > 0, \quad \frac{df}{d\mu} = -\frac{f'}{f''} > 0, \quad \frac{dr}{d\mu} = -1 < 0, \quad \frac{dw}{d\mu} = k > 0. \tag{2.4.18}$$

The effects on the aggregated variables are the same as in the model with homogeneous population in Section 2.2. To determine the impact on m by equation (2.4.15), we notice the distribution policy $\overline{n}_1\mu_1 + \overline{n}_2\mu_2 = \mu$, that is

$$\overline{n}_1\hat{\mu}_1 + \overline{n}_2\hat{\mu}_2 = 1,$$

where $\hat{\mu}_j \equiv d\mu_j/d\mu > 0$. Taking derivatives of equation (2.4.15) with respect to μ yields

$$\frac{dm}{d\mu} = \left(-\frac{1}{f''} + \frac{d\lambda_0}{d\mu}\right)\xi_0 + (k + \lambda_0)\frac{d\xi_0}{d\mu},$$

where

$$\frac{d\lambda_0}{d\mu} = \frac{\lambda_0 k}{w} + \sum_j \frac{n_j\lambda_j^2 z_j w}{(\lambda_j r - \xi_j)^2},$$

$$\frac{d\xi_0}{d\mu} = \frac{\xi_0}{\mu} - \frac{\xi_0^2}{\mu}\left(\sum_j \frac{n_j\xi_j\hat{\mu}_j}{(\lambda_j r - \xi_j)} + \frac{n_j\xi_j\lambda_j\mu_j}{(\lambda_j r - \xi_j)^2}\right).$$

As the terms, $d\lambda_0/d\mu$ and $d\xi_0/d\mu$ are ambiguous (if we do not further specify the parameter values), the impact on m is ambiguous.

From equations (2.4.16), we obtain

$$\frac{dk_j}{d\mu} = \frac{\xi_j\mu_j}{\mu(\lambda_j r - \xi_j)}\frac{dm}{d\mu} + \frac{\xi_j\hat{\mu}_j m/\mu - \xi_j\mu_j m/\mu^2 + \lambda_j k_j - \lambda_j z_j k}{\lambda_j r - \xi_j}.$$

We see that the effects on the wealth distribution are difficult to explicitly judge. As it is straightforward to provide analytical results on the effects of changes for any variable, we will not make further comparative statics analysis.

2.5 A two-sector monetary growth model

Solow's one-sector growth model and Uzawa's two-sector growth model have played the role of the key models in the neoclassical growth theory. These two models and their various extensions and generalizations are fundamental for the

development of new economic growth theories as well. Since Uzawa proposed the model in 1961, many works have been published to extend and generalize the model from the 1960s till today.[23] But all these studies follow the Ramsey approach to consumer behavior. This section proposes another approach to consumer behavior to re-examine the basic issues addressed by the two-sector growth model within the OSG framework.

2.5.1 The two-sector monetary growth model

Like in the traditional two-sector growth model, it is assumed that consumption and capital goods are different commodities, which are produced in two distinct sectors. Labor is homogeneous and labor grows at an exogenously given exponential rate, n, which is assumed to be zero in this section. The assumption of zero population growth rate does not affect our analysis in the sense that we can get similar results with a constant growth rate (because the two sectors exhibit constant returns to scale). Capital depreciates at a constant exponential rate, δ_k, which is independent of the manner of use. The production functions are given by

$$F_j\left(K_j(t), N_j(t)\right), \quad j = i, s,$$

where F_j are the output of sector j, K_j, and N_j are respectively the capital and labor used in sector j, F_j the production functions, the subscripts i and s denote the capital good sector and the consumption good sector. Assume F_j to be neoclassical. We have

$$f_j(t) = f_j\left(k_j(t)\right), \quad f_j(t) \equiv \frac{F_j(t)}{N_j(t)}, \quad k_j(t) \equiv \frac{K_j(t)}{N_j(t)}, \quad j = i, s.$$

The functions f_j have the following properties:

i $f_j(0) = 0$;

ii f_j are increasing, strictly concave on R^+, and C^2 on R^{++}; $f_j'\left(k_j\right) > 0$ and $f_j''(k) < 0$; and

iii $\lim_{k_j \to 0} f_j'\left(k_j\right) = \infty$ and $\lim_{k_j \to +\infty} f_j'\left(k_j\right) = 0$.

Markets are competitive; thus labor and capital earn their marginal products, and firms earn zero profits. We introduce

N: fixed population;
$K(t)$: total capital stocks of the economy at time t;
$c(t)$: the level of per capita consumption of consumption goods;
$P_j(t)$: price of goods $j, j = i, s$, at time t; and
$r(t)$ and $w(t)$: the rate of interest and wage rate in terms of capital goods.[24]

The marginal conditions for the two sectors are

$$r(t)+\delta_k = f_i'(k_i(t)) = p(t)f_s'(k_s(t)),$$
$$w(t) = f_i(k_i(t)) - k_i(t)f_i'(k_i(t)) = p(t)\left[f_s(k_s(t)) - k_s(t)f_s'(k_s(t))\right], \quad (2.5.1)$$

where $p(t) \equiv P_s(t)/P_i(t)$.

The total capital stock is allocated to the two sectors. As full employment of labor and capital is assumed, we have

$$K_i(t)+K_s(t) = K(t), N_i(t)+N_s(t) = N.$$

We rewrite the above equations as

$$n_i(t)k_i(t)+n_s(t)k_s(t) = k(t), n_i(t)+n_s(t) = 1, \quad (2.5.2)$$

where

$$k_j(t) \equiv \frac{K_j(t)}{N_j(t)}, \quad n_j(t) \equiv \frac{N_j(t)}{N}, \quad j = i, s.$$

Money is introduced by assuming that a central bank distributes at no cost to the population a per capita amount of fiat money $M(t) > 0$. The scheme according to which the money stock evolves over time is deterministic and known to all agents. With μ being the constant net growth rate of the money stock, $M(t)$ evolves over time according:

$$\dot{M}(t) = \mu M(t), \quad \mu > 0.$$

It is assumed that the government uses $g(t)$ units of the output good per capita for public purpose

$$g(t) = \frac{\dot{M}(t)}{P_i(t)} = \frac{\mu M(t)}{P_i(t)} = \mu m(t), \quad (2.5.3)$$

where $m(t) \equiv M(t)/P_i(t)$. As equation (2.2.3), we have

$$\frac{\dot{m}(t)}{m(t)} = f_i'(k_i(t)) + \mu - \delta_k. \quad (2.5.4)$$

From $M = mP_i$, we have

$$\pi(t) = \frac{\dot{P}_i(t)}{P_i(t)} = \mu - \frac{\dot{m}(t)}{m(t)}. \quad (2.5.5)$$

The disposable income is given by

$$\hat{y}(t) = y(t) + a(t) - \pi(t)m(t), \quad (2.5.6)$$

where

$$y(t) \equiv r(t)k(t) + w(t), \quad a(t) \equiv k(t) + m(t).$$

The budget constraint is given by

$$p(t)c(t) + s(t) = \hat{y}(t). \tag{2.5.7}$$

We specify the utility function in the Cobb-Douglas form: $U = c^{\xi_0} s^{\lambda_0}$. The optimal solution is

$$c(t) = \frac{\xi \hat{y}(t)}{p(t)}, \quad s(t) = \lambda \hat{y}(t), \tag{2.5.8}$$

where

$$\xi \equiv \frac{\xi_0}{\xi_0 + \lambda_0}, \quad \lambda \equiv \frac{\lambda_0}{\xi_0 + \lambda_0}.$$

The wealth accumulation is given by

$$\dot{a}(t) = s(t) - a(t). \tag{2.5.9}$$

The balance of demand and supply in consumption goods market is given by

$$c(t)N = F_s(t). \tag{2.5.10}$$

We have thus built the economic system.

2.5.2 The dynamics and steady state

We show that the motion of the system is determined by a differential equation. First, from

$$r(t) + \delta_k = f_i'(k_i) = p f_s'(k_s)$$

in (2.5.1), we get

$$p = \frac{f_i'(k_i)}{f_s'(k_s)}.$$

Substituting this equation into $f_i(k_i) - k_i f_i'(k_i) = p\left(f_s(k_s) - k_s f_s'(k_s)\right)$ in equations (2.5.1) yields

$$\Psi_i(k_i) \equiv \frac{f_i(k_i)}{f_i'(k_i)} - k_i = \frac{f_s(k_s)}{f_s'(k_s)} - k_s \equiv \Psi_s(k_s). \tag{2.5.11}$$

From the properties of $f_i(k_i)$, we can show that the function, $\Psi_i(k_i)$, has the following properties

$$\Psi_i(0) = 0,\ \Psi_i(k_i) > 0 \quad \text{for } k_i > 0,\ \Psi_i'(k_i) = -\frac{f_i(k_i)f_i''(k_i)}{f_i'^2(k_i)} \geq 0.$$

The function $\Psi_s(k_s)$ has the same properties in k_s. We see that for any given $k_s \geq 0$, equation (2.5.11) determines $k_i \geq 0$ as a unique function of k_s, denoted by

$$k_i = \Omega(k_s).$$

We have $\Omega(0) = 0$ and $\Omega' > 0$. From equations (2.5.2), we solve the labor distribution as functions of k, k_s, and k_i as follows

$$n_i = \frac{k - k_s}{k_i - k_s}, \quad n_s = \frac{k_i - k}{k_i - k_s}, \tag{2.5.12}$$

where we omit time expression. From equation (2.5.10), we have

$$c = n_s f_s(k_s).$$

From this equation and $c = \xi\hat{y}/p$, we have

$$\left(\frac{k_i - k}{k_i - k_s}\right)\frac{f_s(k_s)f_i'(k_i)}{f_s'(k_s)} = \xi\hat{y},$$

where we use n_s in (2.5.12) and $p = f_i'(k_i)/f_s'(k_s)$. In this study, we omit the case of $k_i = k_s$. Substituting the definition of $\hat{y}$ into the above equation, we solve

$$k = \frac{k_i\Omega_0(k_i) - \xi w - (1-\pi)\xi m}{\xi + \xi r + \Omega_0(k_i)}, \tag{2.5.13}$$

where

$$\Omega_0(k_i) \equiv \frac{f_s f_i'}{(k_i - k_s)f_s'}.$$

From equation (2.5.3), we have

$$\dot{m} = m\tilde{f}(k_i), \tag{2.5.14}$$

where $\tilde{f}(k_i) \equiv f_i'(k_i) + \mu - \delta_k$. We see that the motion of $m(t)$ can be expressed as a function of $k_i(t)$ and $m(t)$. By equations (2.5.5) and (2.5.14), $\pi = \mu - \tilde{f}(k_i)$. Substituting this equation into equation (2.5.13) yields

$$k = \Omega(k_i, m) \equiv \frac{k_i\Omega_0(k_i) - \xi w - \left(1 - \mu + \tilde{f}(k_i)\right)\xi m}{\xi + \xi r + \Omega_0(k_i)}. \tag{2.5.15}$$

We should require $k > 0$. Taking derivatives of equation (2.5.15) with respect to t yields

$$\dot{k} = \Lambda(k_i, m)\dot{k}_i - \frac{\left(1 - \mu + \tilde{f}(k_i)\right)\xi m\tilde{f}}{\xi + \xi r + \Omega_0(k_i)}, \tag{2.5.16}$$

where

$$\Lambda(k_i, m) \equiv \frac{-\Omega\left(-\xi f_i'' + \Omega_0'\right) + \Omega_0^+ k_i \Omega_0' + \xi k_i f_i'' - \xi f_i'' m}{\xi + \xi r + \Omega_0}.$$

Substituting $a = k + m$, $s = \lambda\hat{y}$, and equation (2.5.16) into equation (2.5.9) yields

$$\dot{k}_i = \tilde{\Omega}(k_i, m) \equiv \frac{(\lambda r - \xi)\Omega + \lambda w - \left(\xi + \lambda\mu - \lambda\tilde{f}\right)m}{\Lambda} + \left(\frac{\xi\tilde{f} - \xi\mu - \xi r - \Omega_0}{\xi + \xi r + \Omega_0}\right)\frac{m\tilde{f}}{\Lambda}, \tag{2.5.17}$$

where we also use $\dot{m} = m\tilde{f}(k_i)$ and

$$\hat{y} = (1 + r)k + w + \left(1 - \mu + \tilde{f}\right)m. \tag{2.5.18}$$

We summarize the analytical results in the following lemma.

Lemma 2.5.1
The motion of the economic system is described by two differential equations with $k_i(t)$ and $m(t)$ as variables

$$\dot{k}_i = \tilde{\Omega}(k_i, m),$$
$$\dot{m} = m\tilde{f}(k_i).$$

All the other variables are determined as functions of $k_i(t)$ and $m(t)$ by the following procedure: $k(t)$ by (2.5.15) $\rightarrow \pi = \mu - \tilde{f} \rightarrow a = k + m \rightarrow w$ and r by (2.5.1) $\rightarrow k_s$ by (2.5.11) $\rightarrow f_j\left(k_j\right), j = i, s \rightarrow n_j(t)$ by (2.5.12) $\rightarrow p = f_i'(k_i)/f_s'(k_s) \rightarrow N_j(t) = n_j(t)N \rightarrow F_j = f_j\left(k_j\right)N_j \rightarrow \hat{y}$ by (2.5.18) $\rightarrow c$ and s by (2.5.8).

From equations (2.5.14) and (2.5.17), at steady state we have[25]

$$f_i'(k_i) + \mu - \delta_k = 0,$$
$$(\lambda r - \xi)\Omega + \lambda w - (\xi + \lambda\mu)m = 0, \tag{2.5.19}$$

We require $\delta_k > \mu$. From $f_i' + \mu - \delta_k = 0$, we uniquely determine k_i. Hence, we uniquely determine k_s, r, w, p, f_i, f_s, and Ω_0. From the second equation (2.5.19),

we solve m as

$$m = \frac{\xi w + \lambda w \Omega_0 - (\xi + \lambda\mu) k_i \Omega_0}{(\xi + \lambda\mu)\Omega_0}, \tag{2.5.20}$$

where we use $r = -\mu$. If both k_i and m positive, following Lemma 2.5.1 we determine all the other variables in the dynamic system.[26]

As the expressions are tedious, for convenience of interpretation, we specify $f_j = A_j k_j^{\alpha_j}, 0 < \alpha_j < 1$. It is straightforward to confirm

$$k_s = \alpha k_i, \quad p = \frac{\beta_i A_i}{\beta_s \alpha^{\alpha_s} A_s} k_i^{\alpha_i - \alpha_s}, \quad w = \beta_i A_i k_i^{\alpha_i}, \quad \Omega_0(k_i) = \alpha_1 k_i^{-\beta_i}, \tag{2.5.21}$$

where

$$\alpha_1 \equiv \frac{\alpha \alpha_i A_i}{(1-\alpha)\xi \alpha_s}, \quad \alpha \equiv \frac{\beta_i \alpha_s}{\beta_s \alpha_i}, \quad \beta_j \equiv 1 - \alpha_j, \quad j = i, s.$$

The price of consumer goods is positively related to the technological level of the capital good sector but negatively related to that of the consumer good sector. The price is positively or negatively related to the capital intensity of the capital good sector, depending on the sign of $\alpha_i - \alpha_s$. If $\alpha_i = \alpha_s$, then the price is constant, $p = A_i / A_s$. In the remainder of this section, we require $\alpha_i \neq \alpha_s$. If the equality holds, then labor distribution is invariant in time. The analysis of case $\alpha_i = \alpha_s$ is straightforward.

From $f_i' + \mu - \delta_k = 0$, we solve

$$k_i = \left(\frac{A_i}{\delta_k - \mu} \right)^{1/\beta_i}. \tag{2.5.22}$$

From equations (2.5.21), we get w, p, and k_s. We also solve $r = -\mu$ and $f_j(k_j)$. From $\lambda \hat{y} = a$ and $\hat{y} = w + (1-\mu)a$ (which is obtained from equation (2.5.6) and $r = -\mu$), we solve

$$a = \frac{\lambda w}{\xi + \lambda\mu}, \quad \hat{y} = \frac{w}{\xi + \lambda\mu}. \tag{2.5.23}$$

From $n_i + n_s = 1$, $c = n_s f_s(k_s)$, and $c = \xi \hat{y}/p$, we solve

$$n_i = 1 - n_s, \quad n_s = \frac{\beta_s \xi}{\xi + \lambda\mu}, \tag{2.5.24}$$

where we use equations (2.5.21) and (2.5.23). The labor distribution is determined by the preference, the inflation rate, and technology. From $k_s = \alpha k_i$ and equations (2.5.12) and (2.5.23), we solve

$$k = (n_i + \alpha n_s) k_i, \quad m = \frac{\lambda w}{\xi + \lambda\mu} - (n_i + \alpha n_s) k_i, \tag{2.5.25}$$

where we use $a = k + m$. It is straightforward to show that m is given by

$$m = \frac{[\lambda\beta_i(\delta_k - \mu) - (n_i + \alpha n_s)(\xi + \lambda\mu)]k_i}{\xi + \lambda\mu}. \tag{2.5.26}$$

Hence, when μ and ξ are small we can guarantee a positive m. The conditions say that if the inflation rate is low and the propensity to save is high, the economy has a unique outside money steady state.[27]

2.5.3 *Comparative static analysis*

The effects of the inflation rate

We now examine the effects of the inflation policy on the economic structure with the Cobb-Douglas production functions. From equations (2.5.22) and (2.5.21), we obtain

$$\frac{dk_i}{d\mu} = \frac{k_i}{(\delta_k - \mu)\beta_i} > 0, \quad \frac{dk_s}{d\mu} = \alpha\frac{dk_i}{d\mu} > 0,$$

$$\frac{dp}{d\mu} = \frac{(\alpha_i - \alpha_s)p}{Ak_i}\frac{dk_i}{d\mu}, \quad \frac{dw}{d\mu} = \frac{\alpha_i w}{k_i}\frac{dk_i}{d\mu} > 0.$$

As the inflation rate rises, the capital intensities of the two sectors and the wage rate are increased. The price of consumption goods falls (rises) in the case of $\alpha_i < (>)0$. From equations (2.5.23) and (2.5.24)

$$\frac{da}{d\mu} = \lambda\frac{d\hat{y}}{d\mu} = \left[\frac{\alpha_i}{(\delta_k - \mu)\beta_i} - \frac{\lambda}{\xi + \lambda\mu}\right]a = \frac{a(\alpha_i\xi + \lambda\mu - \beta_i\lambda\delta_k)}{(\xi + \lambda\mu)(\delta_k - \mu)\beta_i},$$

$$\frac{dn_s}{d\mu} = -\frac{dn_i}{d\mu} = -\frac{\lambda n_s}{\xi + \lambda\mu} < 0.$$

We see that if $\alpha_i\xi + \lambda\mu > \beta_i\lambda\delta_k$, then as the inflation rate rises, the total wealth is increased. As the inflation rate rises, some labor will migrate from the consumption goods sector to the capital sector. One of the forces of determining the labor migration direction is that as the inflation rate rises, the rate of interest falls. As the rate of interest falls, the firms tend to use more capital and hence the capital sector tends to employ more workers. From equations (2.5.25), we have

$$\frac{dk}{d\mu} = (n_i + \alpha n_s)\frac{dk_i}{d\mu} - \frac{(\alpha - 1)\lambda n_s k_i}{\xi + \lambda\mu},$$

$$\frac{dm}{d\mu} = \left[\frac{\alpha_i}{(\delta_k - \mu)\beta_i} - \frac{\lambda}{\xi + \lambda\mu}\right]a - \frac{dk}{d\mu}.$$

In general, the effects on the per capita physical wealth and real money balances are ambiguous. Nevertheless, if $\alpha < 1$, that is, $\alpha_s < \alpha_i$ then the per capita physical

wealth is increased. When $\alpha_s > \alpha_i$, the impact on the physical wealth is ambiguous. If the physical wealth is reduced, then the real money balances will be increased. Otherwise, the impact on m is ambiguous.

The productivity of the capital sector

We now examine effects of change in the productivity of the capital sector. Taking derivatives of equations (2.5.22) and (2.5.21) with respect to A_i, we obtain

$$\frac{dk_i}{dA_i} = \frac{k_i}{A_i\beta_i} > 0, \quad \frac{dk_s}{dA_i} = \alpha\frac{dk_i}{dA_i} > 0, \quad \frac{dp}{dA_i} = \frac{\beta_s p}{\beta_i A_i} > 0,$$

$$\frac{dw}{dA_i} = \frac{\alpha_i w}{k_i A}\frac{dk_i}{dA_i} + \frac{w}{A_i} > 0.$$

The capital intensities, wage rate, and price of consumption goods are all increased. From equations (2.5.23) and (2.5.24), we obtain

$$\frac{da}{dA_i} = \lambda\frac{d\hat{y}}{dA_i} = \frac{\alpha_i a}{k_i}\frac{dk_i}{dA_i} > 0, \quad \frac{dn_s}{dA_i} = \frac{dn_i}{dA_i} = 0.$$

The total wealth is increased, but the labor distribution is not affected. As the labor distribution is related to the inflation rate, the preference and the factor substitution parameters in the production function, a change in the total productivity does not affect the labor distribution. From equations (2.5.25), we have

$$\frac{dk}{dA_i} = (n_i + \alpha n_s)\frac{dk_i}{dA_i} > 0, \quad \frac{dm}{d\mu} = \frac{m}{k_i}\frac{dk_i}{d\mu} > 0.$$

The physical wealth and real money balances are increased as the productivity is increased.

The steady state and the propensity to save

We now study effects of the propensity to save. From equations (2.5.22) and (2.5.21), we obtain

$$\frac{dk_i}{d\lambda_0} = \frac{dk_s}{d\lambda_0} = \frac{dp}{d\lambda_0} = \frac{dw}{d\lambda_0} = 0,$$

Change in the preference has no effect on the capital intensities, price and wage rate. From equation (2.5.24), we have

$$\frac{da}{d\lambda_0} = \frac{a\xi}{(\xi + \lambda\mu)\lambda_0} > 0, \quad \frac{d\hat{y}}{d\lambda_0} = \frac{(1-\mu)\xi\hat{y}}{(\xi_0 + \lambda_0\mu)(\xi + \lambda)} > 0,$$

$$\frac{dn_s}{d\lambda_0} = -\frac{dn_i}{d\lambda_0} = -\frac{\mu n_s}{\xi_0 + \lambda_0\mu} < 0.$$

As the propensity to save is increased, some workers change their job from the consumption goods sector to the capital goods sector. The total wealth is increased. From equations (2.5.25), we have

$$\frac{dk}{d\lambda_0} = (n_i + \alpha n_s)\frac{dk_i}{d\lambda_0} - \frac{(\alpha - 1)\mu n_s k_i}{\xi_0 + \lambda_0 \mu}. \quad \frac{dm}{d\lambda_0} = \frac{a\xi}{(\xi + \lambda\mu)\lambda_0} - \frac{dk}{d\lambda_0}.$$

In general, the effects on the total physical wealth and the real money balances are ambiguous. In the case of $\alpha < 1$, the physical wealth is increased.

2.6 Money as a store of value

This chapter develops a few growth models with money and capital in continuous time. It is straightforward to examine the same issues in discrete time. Zhang (2005a: Section 4.8) developed a growth model with money in discrete version. The discrete model is similar to the continuous model in Section 2.2 with regard to the production sector and behaviors of firms. There are many models which deal with interactions between money and growth within the Diamond framework.[28] We may extend the models in this chapter in different directions. For instance, we may synthesize the models in Sections 2.4 and 3.5, examining an economy with multiple sectors and heterogeneous households. As shown later on, it is also possible to introduce endogenous human capital into the model.

APPENDIX

A.2.1 The generalized Tobin model

We now introduce a generalized Tobin model. This section is based on Chapter 6 in Burmeister and Dobell (1970) and Chapter 3 in Zhang (1990). The model is developed and extended within the framework of Tobin's analysis.[29] We assume the presence of a paper currency in addition to a single capital good; wealth may be held in either of these two forms. Money, issued by the government without cost, is assumed to serve as numeraire. Money is desired for current investment and transaction purposes. We assume that the per capita demand for money is a function of per capita money income, per capita money wealth, and the expected yield on capital. The variables, $F(K,N), K, N, f(k)$, and k, are defined as before. The money market is assumed to be always in equilibrium, that is, money demand per capita m^d always equals the actual money supply per capita

$$m^d(t) = G(y(t), w(t), r(t)) = m(t),$$

where m $(= M/N$, where M is money stock, N labor force) is per capita money stock supply, G is a continuous function with regard to the arguments, y $(= pf(k))$ is per capita value of output (where p is price of output in terms of money

as numeraire), $w(=pk+m)$ is per capita money wealth; r is the expected money yield on the capital good. We have

$$r(t)=f'(k(t))-\delta_k+E\left[\frac{dp(t)/dt}{p(t)}\right],$$

where δ_k is a fixed depreciation rate. In the above formation, $E(dp/dt/p)$ is the expected inflation rate. We assume the absence of "money illusion," which means that $G(y,w,r)$ is homogeneous of degree one in its first two arguments.

The real wealth W and real disposable income $Y_d(t)$ are defined as

$$W(t)=K(t)+\psi(t),\ Y_d(t)=F(K(t),N(t))-\delta_k K(t)+\dot{\psi}(t),$$

where $\psi(t)\equiv M(t)/p(t)$. As

$$F(t)=C(t)+\delta_k K(t)+\dot{K}(t),$$

where $C(t)$ is the consumption, we have

$$Y_d(t)=C(t)+\dot{W}(t).$$

That is, real net disposable income is equal to the change in real wealth plus real consumption. It is assumed that real consumption is always a constant fraction of real net disposable income

$$C(t)=cY_d(t),$$

where c is the propensity to consume. This is a flow equilibrium condition satisfied at all points of time. From $Y_d=C+dW/dt$ and $C=cY_d$, we have

$$\dot{W}(t)=sY_d(t),$$

where $s\equiv 1-c$. This implies

$$\dot{K}=s\left(Y+\dot{\psi}-\delta_k K\right)-\dot{\psi}=s(y-\delta_k k)-(1-s)\dot{\psi}, \tag{A.2.1.1}$$

which is referred to as Tobin's fundamental equation.

Taking the derivative of $x=\psi/N$ yields

$$g_x=z-n-g_p,$$

where n is the fixed population growth rate and z is the constant proportional rate of increase in the nominal stock of money. The parameter z is fixed by the government. Let us denote

$$q\equiv E[g_p]=g_p.$$

By $y = f(k)$ and omitting depreciation factors (i.e. $\delta_k = 0$), it is direct to show that we may write (A.2.1.1) and $g_x = z - n - g_p$ as follows

$$\dot{k}(t) = sf(k(t)) - (z - qk(t))cx(t) - nk(t),$$

$$\frac{\dot{x}(t)}{x(t)} = z - n - \frac{\dot{p}(t)}{p(t)}.$$

To complete the system, we have to specify dp/dt. We permit the market for goods and services and the money market to be out of equilibrium and the actual rate of inflation to be different from the expected one. We adopt the Walrasian view that when there is excess demand, the price rises; when there is excess supply, the price falls. Without taking account of expected inflation, we propose the following dynamics

$$\dot{p}(t) = \alpha p(t)[x(t) - g(.)],$$

where α is a positive constant parameter, and the function g is to be specified. It is assumed that the expected inflation rate may be different from the actual inflation rate. The dynamics is specified as

$$\dot{q}(t) = \beta\left[\frac{\dot{p}(t)}{p(t)} - q(t)\right],$$

where β is the so-called "expectation coefficient." This is the "adaptive expectation" equation initially introduced by Cagan (1956). It states that expectations change a constant proportion of the "error" between the actual rate of inflation and the expected one. If $\beta \to \infty$, we again have the perfect foresight equation.

We now come to the problem of specifying the demand function for money. In the case in which the demand for real balances is only for asset purposes, then it is a function of the opportunity cost of holding them, $f'(k) + q$. We now examine the cases in which the two assets of our model, capital and real cash balances are perfect and imperfect substitutes. In the first case, the yields of both assets have to be the same. Otherwise, when $f'(k) + q > 0$ only capital is demanded, and when $f'(k) + q < 0$ only real cash balances are demanded. In this case, we have $g'(.) \to -\infty$, where the derivative is related to $f'(k) + q$. In the second case, we have $f'(k) + q > 0$ because of the obvious superiority of real cash balances. The demand function is negatively related to $f'(k) + q$. Tobin attributed this difference to a risk element involved in asset capital as compared to the risklessness of real cash balances, while Friedman *et al.* attributed this difference to what they call the utility yield of real cash balances. We also have to take account of money which is demanded for transaction purposes. The proxy for per capita transaction demand usually found in the literature is the per capita output $f(k)$; the higher the per capita output, the higher the per capita transaction demand. We thus can generally write g as, $g = G^*[f(k), f' + q]$, or $g = g(k, q)$. In the case of perfect substitutability,

we have $g_k = +\infty$ and $g_q = -\infty$; in the case of imperfect substitutability, $g_k > 0$ and $g_q < 0$.

Summarizing the above discussions, we obtain the following generalized Tobin model

$$\dot{k}(t) = sf(k(t)) - (z - q(t))cx(t) - nk(t).$$

$$\frac{\dot{x}(t)}{x(t)} = z - n - \alpha[x(t) - g(k(t), q(t))].$$

$$\dot{q}(t) = \beta(\alpha\{x(t) - g[k(t), q(t)]\} - q(t)). \tag{A.2.1.2}$$

A positive long-run equilibrium is determined as a solution of

$$sf(k) - c(z - q)x - nk = 0,$$

$$z - n - \alpha\{[x - g(k, q)]\} = 0,$$

$$\alpha[x - g(k, q)] = q.$$

First, we have $x = (sf - nk)/cn$, which exhibits the non-neutrality of money in the sense that the capital–labor ratio of the monetary model is lower than that of the non-monetary one. If $x = 0$, then one has $sf = nk$, which is identical to the solution of the Solow model. If x_0 is positive, then $f/k > n/s$. Non-neutrality follows. As shown by Zhang (1990), the following theorem is held for the Tobin model.

Theorem A.2.1.1

i If both α and $\beta \to \infty$, then the long-run model is locally unstable

ii Even if either α or $\beta \to \infty$, if money is a perfect substitute for capital, the long-run model is locally unstable.

As shown from the comparative analysis by Hadjimichalakis (1971a, 1971b), if the equilibrium is stable, an increase in the rate of change of the nominal quantity of money increases the long run capital intensity and the expected rate of inflation. The following theorem is proved by Benhabib and Miyao (1981).

Theorem A.2.1.2

The equilibrium point is locally asymptotically stable if the following conditions are satisfied

$$\left\{\frac{1}{\alpha x} - \frac{q_0 g_q}{(g - n)g}\right\}\beta + \frac{(1 - s)n}{\alpha k} \leq 1, \quad \frac{kg_k}{g} \geq 1.$$

The theorem says that the smaller the value of β or the greater the value of α, the more likely is stability. Also, the smaller the elasticity of the money demand function with respect to q, or the greater the elasticity with respect to k, the more likely is stability.

The above two theorems imply that the equilibrium may be either stable or unstable. For instance, if we move from adaptive expectations towards perfect

foresight, saddle-point instability may appear as it does in the Tobin model. For the sake of illustration, consider an increase in the stock of money at the equilibrium. The immediate impact of this is to increase the price level while the real money stock tends to fall back to its original level; but the initial increase in the stock of money also tends to increase price expectations and reduce the capital stock. The latter two effects reinforce the decrease in the money supply and may cause the money stock to overshoot its long-run equilibrium. As the money supply keeps falling beyond its equilibrium level, the effects on the two variables are reversed: the capital stock rises and expectations fall. Combined with the direct effect of the money stock on the accumulation of money, the fall of the money stock will now be reversed. This discussion hints at the possibility of oscillations in the long run. In fact, the existence of Hopf bifurcations in the generalized Tobin model has been identified by Benhabib and Miyao (1981). Their results can be summarized as follows.

Theorem A.2.1.3
If there exist a set of parameter values which guarantee the stability of the equilibrium, we can find a value of β, denoted by β_0, such that the Jacobian of the system has a pair of purely imaginary eigenvalues. Moreover, there exists a continuous function $\nu(\varepsilon)$ $(\nu(0) = 0)$ of a parameter ε such that when ε is sufficiently small, the generalized Tobin model has a continuous family of periodic solutions $(k(t, \varepsilon), x(t, \varepsilon), q(t, \varepsilon))^T$, which collapse to the equilibrium point as $\varepsilon \to \infty$.

This theorem identifies the existence of regular oscillations in the system. Such oscillations will continue permanently if the stability of the cycles can be identified. Non-equilibrium economic development is no longer a short-run phenomenon. This theorem shows that the loss of stability that occurs as expectations adjust is associated with the emergence of bounded, persistent oscillations in prices, output and expectations. This holds no matter how quickly prices adjust since there always exists a value of β at which the stability of the equilibrium is lost. The generalized Tobin model can be applied to explain business cycles. Zhang actually improved the results obtained by Benhabib and Miyao in the following aspects: (i) to find stability conditions of the cycle; (ii) to explicitly interpret the parameter h; (iii) to find the explicit expression of the cycle; and (iv) to discuss whether the Hopf bifurcation is supercritical or subcritical. We will not introduce Zhang's results for technical reasons. Supercritical bifurcations mean that if the bifurcation parameter β is increased, the system is stabilized, while if it is decreased, the system becomes unstable and bifurcations may take place. Zhang's results give a complete description of the Hopf bifurcations near the equilibrium. At the equilibrium point, the system is very sensitive to changes of the parameter β. Even when perturbations in β are sufficiently small, structural changes take place, resulting in limit cycles.

A.2.2 Competitive business cycles with money and capital

We now provide a model of long-run competitive two-periodic OLG model with money and capital.[30] A single good is produced and each agent lives two periods.

The young generation sells one unit of labor at a real wage $w(t)$, consumes quantity $c_1(t)$, saves the real quantity, $s(t)$, by holding money and capital. Let $r(t+1)$ denote the real rate of interest between t and $t+1$. The old generation spends all its savings from the previous period on consumption, $c_2(t)$. A typical consumer maximizes his utility subject to the budget constraints as follows

$$\begin{aligned} &MaxU(c_1(t), c_2(t)) \\ &\text{s.t.: } c_1(t) + s(t) \leq w(t), \\ &\qquad c_2(t) \leq r(t+1)(t), c_i(t) \geq 0, \quad i = 1, 2. \end{aligned}$$

Under standard assumptions, the consumer's decision problem has a unique solution characterized by the saving function $s(w(t), r(t+1))$ which satisfies $0 < s(w, r) < w$.

Production is made through a neoclassical constant return to scale technology. Output per capita , $y(t)$, is a function of capital intensity, $k(t)$, i.e. $y(t) = f(k(t))$, where f is increasing, strictly concave on R_+ and C^2 on R_{++}, and satisfies: $\lim_{k\to\infty} f'(k) \in [0, 1[$, $\lim_{k\to 0} f'(k) = \infty$, $kf'(k)$ is non-decreasing, and

$$\lim_{k\to\infty} f(k) - kf'(k) = +\infty, \lim_{k\to 0} f(k) - kf'(k) = 0.$$

Obviously, if $f = Ak^{\alpha}, 0 < \alpha < 1$, the above properties of f are satisfied. The CES production functions with an elasticity of substitution larger than 1 also have these properties. We also have

$$\begin{aligned} &r(t) = f'(k(t)), \\ &w(t) = f(k(t)) - k(t)f'(k(t)) \equiv W((k(t))). \end{aligned}$$

It is known that W is C^1 on R_{++}, increasing, and maps R_+ into itself.

At date 0, the economy is endowed with a fixed quantity of capital k_0 and a quantity of money M. Let $p(t)$ and $m(t)$ $(M/p(t))$ denote respectively the price of goods and the real money. A perfect foresight equilibrium is a sequence of $\{p(t)\}_{t\geq\infty}, \{k(t)\}_{t\geq\infty}, \{r(t)\}_{t\geq\infty}$ and $\{w(t)\}_{t\geq\infty}$, which achieves a competitive temporary equilibrium with perfect foresight at each date. A perfect foresight equilibrium is described by a sequence of $\{k(t)\}_{t\geq\infty}$ and $\{m(t)\}_{t\geq\infty}$ such that

$$m(t) + k(t+1) = S\left(W(k(t)), f'(k(k+1))\right), \tag{A.2.2.1}$$

$$m(t+1) = f'(k(t+1))m(t), k(t) > 0, m(t) \geq 0, k_0 \text{ given.} \tag{A.2.2.2}$$

Equation (A.2.2.1) equalizes the demand and supply of assets; equation (A.2.2.2) equalizes the interest rate on money $p(t)/p(t+1)$ and interest rate on capital $f'(k(t+1))$. We call an equilibrium non-monetary or monetary according to $m_0 = 0$ or $m_0 > 0$. By (A.2.2.1) and (A.2.2.2), we can replace (A.2.2.1) with

$$m(t+1) + k(t+1)f'(k(k+1)) = S\left(W(k(t)), f'(k(k+1))\right)f'(k(k+1)).$$

It can be shown that the above equation determines a backward dynamics as

$$k(t) = \Lambda(k(t+1), m(t+1)) \tag{A.2.2.3}$$

According to the properties of the saving and production functions, it can be proved that Λ is C^1, increasing in each of its arguments, and Λ tends to 0 (resp. infinity) when k goes to 0 (resp. infinity) with the fixed m. A perfect foresight equilibrium can be described by a two-dimensional (backward) mapping

$$\begin{aligned}(k(t), m(t)) &= \left(\Lambda(k(t+1), m(t+1)), \frac{m(t+1)}{f'(k(t+1))}\right) \\ &\equiv \Omega(k(t+1),\ m(t+1)).\end{aligned} \tag{A.2.2.4}$$

We may express the system in vector form as $x(t) = \Omega(x(t+1))$, where $x \equiv (k, m)$ Here, we are interested in a monetary economy. A steady state (k^*, m^*) is determined by $f'(k^*) = 1$ and $m^* + k^* = s(W(k^*), 1)$. We assume $s(W(k^*), 1) > k^*$. Then the conditions for the existence of a unique monetary steady state are satisfied. Under this assumption, there exists at least one inefficient non-monetary steady state $(k > k^*)$[31] and we denote the lest capital intensive one by k_s, i.e. $k_s = \inf\{k > k^* \mid \Lambda(k, 0) = k\}$.

It can be proved that all the periodic orbits must belong to an C^1 invariant curve. This will enable us to focus the dynamical analysis on the curve in one-dimension.

Theorem A.2.2.1

There exists a compact set $K \in R^2_{++}$ and a function h, decreasing and C^1, from R_{++} to R_{++} such that if we define the sets

$$\Gamma = \{(k, m) \in R^2_+ \mid k = h(m)\}, \quad \Gamma_+ = \{(k, m) \in R^2_+ \mid k > h(m)\},$$
$$\Gamma_- = \{(k, m) \in R^2_+ \mid k < h(m)\},$$

then $\{\Gamma, \Gamma_+, \Gamma_-\}$ is an invariant partition of R^2_{++} and

$$x \in \Gamma_+ \Leftrightarrow \lim_{t\to\infty} m(t, x) = \infty, \quad x \in \Gamma_- \Leftrightarrow \lim_{t\to\infty} m(t, x) = 0,$$

$$x \in \Gamma \Leftrightarrow m(t, x) \in K \text{ for } t \text{ being sufficiently large.}$$

The proof is referred to Jullien. The real quantity of money increases infinitely along the orbit of a point greater than x^* and decreases to 0 along the orbit of a smaller point. Between these two behaviors it is possible to exhibit a set of points with an orbit bounded away from 0 and infinity. It can be shown that all cycles must belong to the curve Γ.

Introduce a one-dimensional map: $\phi : R_{++} \to R_{++}$, $m \to m/f'(h(m))$. The map, ϕ, is C^1. The cycles of ϕ are equivalent to the cycles of in the senses of: $\Omega^p(x) = x \Leftrightarrow \phi^m(m) = m$. The following theorem is referred to Jullien.

Theorem A.2.2.2
A sufficient condition for the existence of a cycle of order 2 is

$$S(W(k^*),1) - k^* + 2S_R(W(k^*),1) + 2k^* S_W(W(k^*),1) - \frac{2}{f''(k^*)} < 0.$$

The proof is to verify the existence of solutions of $\phi^2(m) = m$. It should be noted that Jullien also discusses conditions for the existence of periodic points of period 3, which guarantees economic chaos.

3 Money in utility and production functions

The money in the utility function (MIUF) approach was used initially by Patinkin (1965), Sidrauski (1967a, 1967b) and Friedman (1969). In this approach, money is held because it yields some services and the way to model it is to enter real balances directly into the utility function.[1] Sidrauski (1967a) made a benchmark contribution to monetary economics, challenging Tobin's non-neutrality result. He describes his own purpose as follows:

> What differentiates this product is the fact that, in line with Patinkin's presentation of the neoclassical theory of money, and with the classical Fisherian theory of saving, it is based on an explicit analysis of individual's saving behavior, viewed as a process of wealth accumulation aimed at maximizing some intertemporal utility function.

He proposed a framework that explicitly allows for an endogenous treatment of saving behavior. His analytical framework is developed with Patinkin's idea of ensuring a well-defined demand function for money by assuming that the agent's utility is directly affected by money. Sidrauski found that money is superneutral in steady state comparison and changes in the inflation rate have no effect on all the real variables in the economy.[2] Nevertheless, it has become evident that Sidrauski's results are dependent on the specific set-up of the model. For instance, the choice of Ramsey's version of an infinite horizon economy are essential for money to be superneutral. Moreover, the superneutrality in Sidrauski's model is no more held if leisure is introduced into the utility function. As observed by Wang and Yip (1992), the direction of the non-superneutrality result is related to the signs of the cross-partial derivatives of the utility function with respect to consumption, leisure, and real balances. Orphanides and Solow (1990: 225) reveal

> The main lessons were thus already implicit in the work of Tobin and Sidrauski. For those who can bring themselves to accept the single-consumer, finite-horizon, maximization model as a reasonable approximation to economic

life, superneutrality is a defensible presumption. All others have to be ready for a different outcome.

Rather than following the Ramsey approach, this chapter introduces money into the utility function proposed in Chapter 1 to show interactions between money and economic growth. Before introducing money into the OSG framework, we review the Sidrauski model and its various extensions.

3.1 The Levhari and Patinkin approach

We described Tobin's model in the previous chapter. Tobin was concerned with the question about how capital accumulation and production can be affected by actions that merely change the rate of growth of the stock of nominal money. If such effects exist, will they be permanent or merely transitory? Different rates of growth of the money supply will affect different rates of inflation; but are there any real effects in the long run? Tobin examined these questions in a descriptive model with aggregate saving depending only on current income, and seigniorage distributed in such a way as to preclude any distributional effect. In the Tobin model, it predicts a positive relationship between the money growth rate and the levels of capital stocks and output at a steady state. Faster inflation makes households to shift their portfolios in favor of real capital. The equilibrium capital intensity in Tobin's monetary model is lower than that in Solow's real model. As pointed out by Levhari and Patinkin (1968), this conclusion seems unreasonable. If the result of introducing money into an economy is to reduce the capital intensity and hence per capita output and consumption, money should not be introduced. According to Levhari and Patinkin, though equation $M^d/P = \varepsilon F$ describes a positive demand for real money balances in the Tobin model, it does not really provide a rationale for holding of these balances. Levhari and Patinkin argue that to provide a rationale for holding of money balances, money balances should be interpreted either as a consumer's good or as a producer's good. In the first case, money balances should appear in the utility function, and in the second case money should appear in the production function. Before developing our monetary model with capital accumulation in the MIUF approach, we will introduce Levhari and Patinkin's monetary model with money as a consumer's good.[3]

Money is now treated as a consumer good. This implies that money enters the utility function and its services are included in disposable income. Money services are valued at the alternative cost at the margin of holding money balances which is equal to $r+\pi$, as discussed in the Tobin model in the previous chapter. The definition of disposable income now becomes[4]

$$Y_D = Y + (\mu - \pi)m + (r+\pi)m = Y + (r+\mu)m,$$

where $m \equiv M/P$. The difference between the disposable income in this model and that in the Tobin model is $(r+\pi)m$. That is, the MIUF approach defines

real disposable income as net national income plus the real value of the increase in the nominal quantity of money plus the imputed real interest on real money balances. Total consumption is now given by $(1-s)(Y+\mu m+rm)$. As what should be deducted from total commodity output in order to obtain investment in commodities is not total consumption, but physical consumption, which is given by

$$(1-s)(Y+(\mu+r)m)-(r+\pi)m.$$

Corresponding to the capital accumulation equation and money demand function in the Tobin model, we now have

$$\dot{K}=F-(1-s)(F+(\mu+r)m)+(r+\pi)m,$$

$$\frac{M^d}{P}=\varepsilon F=m.$$

This is the MIUF model by Levari and Patinkin. The equilibrium condition is given by

$$s-(1-s)\varepsilon n+s\varepsilon(\pi+r)=\frac{nk}{f(k)},$$

in which we use

$$\frac{d(m/N)}{dt}=\mu-\pi-n=0,$$

for determining π as a constant with n and μ as parameters. As the Tobin equilibrium condition is given by

$$s-(1-s)n\varepsilon=\frac{nk}{f(k)},$$

It can be seen that the equilibrium value of k in the MIUF approach is higher than that in the Tobin model. The approach by Levhari and Patinkin did not explicitly use utility function. We now introduce the Sidrauski model and some of its variations, which explicitly introduce money into utility function.

3.2 The Sidrauski model and its variations

Section 2.1 described the Tobin model with capital accumulation. The model assumes a constant saving rate. With regard to this approach, Sidrauski (1967a) comments

> The major limitation this analysis is given by the fact that we have postulated a saving function and a demand function for real cash balances that are not

explicitly derived from the maximizing behavior of the individual economic units of the economy. A more reasonable procedure would be to consider a representative economic unit in this simplified world and to analyze first how this economic unit would behave in a monetary economy if it attempted to maximize its well-being over its economic horizon.

Sidrauski (1967a) provided the first formation of a monetary growth in the Ramsey approach.[5] The household is to maximize the following utility functional

$$U = \int_0^\infty u(c(t), m(t))e^{-\rho t}dt,$$

where $c(t)$ and $m(t)$ are per capita consumption and real money balances and ρ is the rate of time preference.[6] Real per capita accumulation is given by

$$\dot{a}(t) = f(k(t)) + x(t) - c(t) - na(t) - \pi(t)m(t), \tag{3.2.1}$$

where $a(t) \equiv k(k) + m(t)$ and $x(t)$ is the government transfers. The problem is to maximize the objective subject to constraint (3.2.1).

The present value Hamiltonian is

$$\mathrm{H} = u(c, m) + \lambda[f(k) + x - c - na - \pi m] + \bar{\lambda}(a - k - m),$$

where $\lambda(t)$ and $\bar{\lambda}(t)$ are respectively the costate variable associated with the flow constraint and the Lagrangian multiplier associated with the stock constraint. Necessary conditions for an internal solution are

$$\begin{aligned}
&\dot{\lambda} = (\rho + n)\lambda - \bar{\lambda},\\
&\lambda f_k(k) - \bar{\lambda} = 0,\\
&u_m - \lambda\pi - \bar{\lambda} = 0,\\
&u_c - \lambda = 0.
\end{aligned} \tag{3.2.2}$$

From equations (3.2.2), it is straightforward to show that the motion of $\lambda(t)$ can also be expressed as

$$\dot{\lambda} = (\rho + n - f_k(k))\lambda.$$

An equilibrium point of the economy is given by setting $\dot{\lambda} = 0$. By the above equation, we have

$$f_k(k) = \rho + n. \tag{3.2.3}$$

From equation (3.2.3), we conclude that the equilibrium value of k is only dependent on technology, the preference parameter, ρ, and the population growth rate, n. Different from the Tobin model, the Sidrauski model predicts that the equilibrium value of k is independent of inflation and the rate of money growth. Money is superneutral in the Sidrauski model.[7]

Empirical studies provide ambiguous conclusions about superneutrality.[8] In fact, the superneutrality of money is not a general result even within the Ramsey framework. Some models originated from the Sidrauski model do not have the property. For instance, we now consider a dynamic model with money in utility function, proposed by Turnovsky.[9] The representative agent now maximizes the following utility function

$$U = \int_0^\infty u(c(t), T(t), g(t), m(t))e^{-\rho t}dt,$$

where $T(t)$ is the work time and $g(t)$ is the government real expenditure. The budget constraint is given by

$$c(t) + \dot{k}(t) + \dot{b}(t) = F(k(t), \quad T(t)) + r(t)b(t) - \pi(t)m(t) - \tau(t), \tag{3.2.4}$$

where $b(t)$ is the real stock of government bonds and $\tau(t)$ is the tax. It can be shown that the optimization leads to the following equilibrium

$$\begin{aligned} &U_c = \lambda, \quad U_T = -\lambda F_T(k, T), \\ &\frac{U_m}{\lambda} - \pi = r = F_k(k, T), \\ &\frac{\dot{\lambda}}{\lambda} = \rho + n - F_k(k, T). \end{aligned} \tag{3.2.5}$$

The model is completed with the assumption that the monetary policy is given by $\dot{M} = \mu M$, where μ is fixed. It is straightforward to analyze properties of the model.[10] It can be seen that if the work time does not enter the production function and utility function, then money is superneutral. Nevertheless, in general the capital intensity is affected by the rate of monetary growth.

There are many economic growth models with capital and/or knowledge. It is straightforward to introduce money into these growth models. For instance, a recent extension of the Sidrauski model is given by Ho *et al.* (2007). The model is a combination of Turnovsky's model and growth models with production externalities with differently chosen government policies. The production function with externalities is specified as

$$F(t) = Ak^\alpha(t)T^{1-\alpha}(t)\overline{K}^\psi(t), \quad 0 < \alpha < 1, \quad 0 < \psi \leq 1 - \alpha,$$

where α, ψ, and A are parameters and $\overline{K}(t)$ is the average capital.[11] The wage rate and rate of interest rate are given by

$$w(t) = (1-\alpha)\frac{F(t)}{T(t)}, \quad r(t) = \alpha\frac{F(t)}{k(t)},$$

with $k(t) = \overline{K}(t)$. The household is to maximize

$$U = \int_0^\infty u(c(t), T(t), m(t))e^{-\rho t}dt,$$

subject to[12]

$$(1+\tau_c)c(t) = F(t) - \dot{k}(t) - \dot{m}(t) - \pi(t)m(t),$$

in which τ_c is consumption tax rate. The government uses a consumption tax and money issuing to finance a government purchase, $G(t)$, that is not valued by private agents. Assume

$$G = \beta F,$$

where $\beta(< 1)$ is a positive parameter. The government's budget is balanced at each point of time[13]

$$\tau_c c(t) + \dot{m}(t) + \pi(t)m(t) = G(t) = \beta F(t).$$

Let μ stand for nominal money growth rate. Ho *et al.* examine the effects of different combinations of parameters, such as, τ_c and μ, upon the economic system. In general, money is non-superneutral. As few new insights can be obtained from the standard but tedious analysis, we will not examine the behavior of the model.

3.3 The one-sector MIUF-OSG model

This section develops a model similar to the model in Section 2.1, except that we introduce money to utility function. The only difference is that in this section money is held because it directly affects utility function, while in Section 2.1 money is held as a store of value.

3.3.1 The MIUF-OSG dynamics

The variables, $N(=1), K(t), F(K(t), N), y(t) = f(k(t)), k(t), \mu, \delta_k, M(t)$, and $m(t)$ are defined as in Section 2.2. Suppose that the economy contains two outside

assets: fiat money and physical capital. The marginal conditions for the production sector are

$$r + \delta_k = f'(k), \quad w = f(k) - kf'(k). \tag{3.3.1}$$

Let $\mu, M(t)$, and $m(t)$ be defined as in Section 2.2. With μ being the constant net growth rate of the per-capita money stock, $M(t)$, evolves over time according to

$$\dot{M}(t) = \mu M(t), \quad \mu > 0.$$

The government expenditure in real terms per capita, $\tau(t)$, is given by

$$\tau(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t). \tag{3.3.2}$$

The representative household receives $\mu m(t)$ units of paper money from the government through a "helicopter drop," also considered to be independent of his money holdings.

The disposable income in real terms, $\hat{y}$, is

$$\hat{y} = rk + w - \pi m + \tau + a, \tag{3.3.3}$$

where $a(t) \equiv k(t) + m(t)$.[14] We assume that at each point of time the consumer's utility function for holding money, consuming goods and saving is being represented by the following utility function[15]

$$U(t) = m^{\varepsilon_0}(t)c^{\xi_0}(t)s^{\lambda_0}(t), \quad \varepsilon_0, \xi_0, \lambda_0 > 0,$$

where ε_0 is called propensity to hold money, ξ_0 propensity to consume, and λ_0 propensity to own wealth. The "price" of holding money is $1 + r(t)$. The budget constraint is given by

$$(1 + r(t))m(t) + c(t) + s(t) = \hat{y}(t).$$

Insert equation (3.3.4) into the budget constraint[16]

$$(r + \pi)m + c + s = y_a \equiv f + \delta k + \tau, \tag{3.3.4}$$

where we use $f = rk + w + \delta_k k$ and $\delta \equiv 1 - \delta_k$. Consumers' problem is to choose current money, consumption, and savings in such a way that utility levels are maximized. Maximizing $U(t)$ subject to the budget constraint (3.3.4) yields

$$(r + \pi)m = \varepsilon y_a, \quad c = \xi y_a, \quad s = \lambda y_a, \tag{3.3.5}$$

where

$$\varepsilon \equiv \rho\varepsilon_0, \quad \xi \equiv \rho\xi_0, \quad \lambda \equiv \rho\lambda_0, \quad \rho \equiv \frac{1}{\varepsilon_0 + \xi_0 + \lambda_0}.$$

The expenditure spent on "consuming money," $(r+\pi)m$, is proportional to the potential disposable income, $y_a(t)$, and the (relative) propensity to use money. We see that $m(t)$ is negatively related to $r+\pi$. This relation is assumed in the Tobin model and the Levhari and Patinkin's monetary model as mentioned before. In our approach, this relation results from optimizing behavior of households.

The real wealth changes as follows

$$\dot{a}(t) = s(t) - a(t). \tag{3.3.6}$$

We have thus built the model.

3.3.2 The dynamics and steady state

First, we need to find equations which describe motion of the dynamic system. From $\pi = \mu - \dot{m}/m$ and $(r+\pi)m = \varepsilon y_a$, we obtain

$$\dot{m} = (r+\mu-\varepsilon\mu)m - \varepsilon f - \varepsilon\delta k, \tag{3.3.7}$$

where we use $\tau = \mu m$. As r and w are functions of k, the right hand side of equation (3.3.7) is a function of k and m.

Substituting $s = \lambda\bar{y}$ and the definition of a and equation (3.3.4) into equation (3.3.6), we have

$$\dot{k} = \lambda f - (1-\lambda\delta)k - (1-\lambda\mu)m - \dot{m}. \tag{3.3.8}$$

We see that $\dot{k}(t)$ is a function of $k(t)$ and $m(t)$. The dynamics consists of two differential equations, (3.3.7) and (3.3.8), with two variables, $k(t)$ and $m(t)$.

We now examine dynamic properties of the differential equations. From equations (3.3.7) and (3.3.8), a steady state is determined by

$$\begin{aligned} &(r+\mu-\varepsilon\mu)m - \varepsilon f - \varepsilon\delta k = 0, \\ &\lambda f - (1-\lambda\delta)k - (1-\lambda\mu)m = 0. \end{aligned} \tag{3.3.9}$$

From the last equation of (3.3.9), we solve m as a function of k as follows

$$m = \frac{\lambda f - (1-\lambda\delta)k}{1-\lambda\mu}. \tag{3.3.10}$$

As $1-\lambda\delta > 0$ and $1-\lambda\mu$, and we should require $m > 0$, we should require $f/k > (1-\lambda\delta)/\lambda$. It is straightforward to confirm that there is a unique value of k, denoted as k^*, such that $f(k^*)/k^* = (1-\lambda\delta)/\lambda$, that is, $m = 0$. As f/k falls in k, we should require $0 < k < k^*$. Moreover, we should require $r+\mu > 0$ at steady

state, that is, $f'(k) - \delta_k + \mu > 0$. To guarantee this inequality, we should require $\delta_k > \mu$. There is a unique value of k, denoted as k°, such that

$$f'(k^\circ) - \delta_k + \mu = 0.$$

Introduce $k^c \equiv \min\{k^\circ, k^*\}$. Then we should require $0 < k < k^c$. Substituting equation (3.3.10) into the first equation in (3.3.9) yields

$$\Omega(k) \equiv (r + \mu - \varepsilon\mu)\frac{\lambda f - (1 - \lambda\delta)k}{1 - \lambda\mu} - \varepsilon f - \varepsilon\delta k = 0. \tag{3.3.11}$$

It is straightforward to confirm

$$\Omega(k^c) = \begin{cases} -\varepsilon f(k^*) - \varepsilon\delta k^* < 0, \text{ if } k^c = k^*, \\ -\varepsilon\mu k^\circ \dfrac{\lambda f(k^\circ)/k^\circ - (1 - \lambda\delta)}{1 - \lambda\mu} - \varepsilon f(k^\circ) - \varepsilon\delta k^\circ < 0, \quad \text{if } k^c = k^\circ, \end{cases}$$

$$\Omega(0) > 0, \tag{3.3.12}$$

where we use $r = f'(0) - \delta_k \to +\infty$ and require $kf'(k) \to 0$ and $f(k)f'(k) \to \infty$[17] as $k \to 0$. Hence, the equation has at least one solution for $0 < k < k^c$. We have

$$\Omega' = mf'' + (r + \mu - \varepsilon\mu)\frac{\lambda r - (1 - \lambda)}{1 - \lambda\mu} - \varepsilon f' - \varepsilon\delta.$$

As mf'', $-\varepsilon f'$ and $-\varepsilon\delta$ are negative, we see that if $(r + \mu - \varepsilon\mu)(\lambda r - 1 + \lambda) < 0$, then $\Omega' < 0$.

We rewrite Ω' as

$$\Omega' = mf'' + \left(r + \mu - \varepsilon\mu - \frac{\varepsilon}{\lambda}\right)\lambda f' - (r + \mu)\frac{(1 - \lambda\delta)}{1 - \lambda\mu} - \varepsilon\left[\frac{\delta - \mu}{1 - \lambda\mu}\right].$$

As the term, $r + \mu - \varepsilon\mu - \varepsilon/\lambda$, may be either positive or negative, the sign of Ω' is ambiguous. If this sign is negative, then we can guarantee the uniqueness of steady state. At any steady state, the two eigenvalues, ϕ_j, are given by

$$\phi^2 - (m_1 - m_2 + m_3)\phi + (1 - \lambda\mu)m_2 + m_3 m_1 = 0,$$

where

$$m_1 \equiv r + \mu - \varepsilon\mu > 0, \quad m_2 \equiv f''m - \varepsilon f' - \varepsilon\delta < 0, \quad m_3 \equiv \lambda f' - (1 - \lambda\delta).$$

It is straightforward to confirm that irrespective of the sign of m_2, the real part of at least one eigenvalue is positive.

Proposition 3.3.1
The system has at least one steady state. If it is unique, it is locally unstable.

To further examine whether the system has a unique steady state, we let $f = Ak^{\alpha}$. According to the definitions, we have

$$k^* = \left(\frac{\lambda A}{1-\lambda\delta}\right)^{1/\beta}, \quad k^\circ = \left(\frac{\alpha A}{\delta_k - \mu}\right)^{1/\beta}. \tag{3.3.13}$$

The equilibrium condition, (3.3.11), can now be written in the form of

$$x^2 + ax - b = 0, \tag{3.3.14}$$

where

$$x \equiv k^{\beta}, \quad \alpha_1 = \frac{\alpha A}{1-\lambda\mu} > 0, \quad \alpha_2 = \frac{\mu - \varepsilon\mu}{1-\lambda\mu} > 0,$$
$$a \equiv \frac{(1-\lambda\delta)\alpha_1 + \varepsilon A - \lambda A\alpha_2}{(1-\lambda\delta)\alpha_2 + \varepsilon\delta}, \quad b \equiv \frac{\lambda A\alpha_1}{(1-\lambda\delta)\alpha_2 + \varepsilon\delta} > 0.$$

The problem has a unique solution as

$$k = \left(\frac{-a + \sqrt{a^2 + 4b}}{2}\right)^{1/\beta}. \tag{3.3.15}$$

As $\Omega(k) = 0$ has at least one solution for $0 < k < k^c$, equation (3.3.14) has only one positive solution. We conclude k in equation (3.3.15) satisfies $0 < k < k^c$, which k^c is defined in (3.3.13).

Corollary 3.3.1
If the production function is specified in the Cobb-Douglas form, the dynamic system has a unique steady state. The steady state is locally unstable.

3.3.3 Effects of changes in inflation, productivity, and preference

This section examines effects of changes in some parameters upon the steady state. It is straightforward to confirm that at any steady state we have

$$\frac{d\Omega}{dk} = mf'' - \frac{(f - f'k)\varepsilon}{\lambda f - (1-\lambda\delta)k} < 0. \tag{3.3.16}$$

This is important for judging the effects of changes in the parameters.

The inflationary policy

First, we are concerned with effects of change in μ. Taking derivatives of (3.3.11) with respect to μ yields

$$-\frac{d\Omega}{dk}\frac{dk}{d\mu}=\left[(1-\varepsilon)+\frac{r+\mu-\varepsilon\mu}{(1-\lambda\mu)}\right]m>0. \tag{3.3.17}$$

A rise in μ increases the per capita physical wealth. As the capital intensity rises, the rate of interest falls and per capita output level and wage rate rise. The Tobin effect prevails, like in the model in Section 2.1. By equations (3.3.10), (3.3.4), and (3.3.5), we calculate

$$\frac{dm}{d\mu}=\frac{\lambda m}{1-\lambda\mu}+\left[\frac{\lambda f'-(1-\lambda\delta)}{1-\lambda\mu}\right]\frac{dk}{d\mu},$$

$$\frac{da}{d\mu}=\frac{dk}{d\mu}+\frac{dm}{d\mu}=\frac{\lambda m}{1-\lambda\mu}+\left[\frac{\lambda f'+\lambda\delta-\lambda\mu}{1-\lambda\mu}\right]\frac{dk}{d\mu}>0,$$

$$\frac{dy_a}{d\mu}=\frac{1}{1-\lambda\mu}+\left[f'+\frac{\lambda\mu f'+\delta-\mu}{1-\lambda\mu}\right]\frac{dk}{d\mu}>0,$$

$$\frac{1}{\xi}\frac{dc}{d\mu}=\frac{1}{\lambda}\frac{ds}{d\mu}=\frac{dy_a}{d\mu}>0,$$

where we require $\delta_k+\mu<1$ in judging the sign of a and $\bar{y}$. The impact on the real money holdings is ambiguous. The total wealth and consumption level are increased. Although the Tobin effect prevails in this model and the model with money as a store of value in Section 2.1, an increase in μ raises the total wealth and consumption level in the MIUF model, but reduces the total wealth and consumption level in the model in Section 2.1. It can be seen that the effects of an increase in μ are all positive for the real economy. It should be remarked that μ cannot be increased arbitrarily in this model as $\delta_k>\mu$ is required for guaranteeing the existence of the steady state.

Improvement in the total productivity

To examine effects of technological change, we specify $f=Ak^{\alpha}$, where $0<\alpha<1$. Taking derivatives of (3.3.11) with respect to A yields

$$-\frac{d\Omega}{dk}\frac{dk}{dA}=\left(\frac{\lambda f-(1-\lambda\delta)k}{1-\lambda\mu}\right)\frac{r}{A}+\left[\frac{k}{\lambda f-(1-\lambda\delta)k}\right]\frac{\varepsilon f}{A}>0. \tag{3.3.18}$$

A rise in A increases the per capita wealth. As capital intensity rises, the rate of interest falls and per capita output level and wage rate rise. By equations (3.3.10),

(3.3.4) and (3.3.5), we calculate

$$\frac{dm}{dA} = \frac{\lambda f}{(1-\lambda\mu)A} + \left[\frac{\lambda f' - (1-\lambda\delta)}{1-\lambda\mu}\right]\frac{dk}{dA},$$

$$\frac{da}{dA} = \frac{dk}{dA} + \frac{dm}{dA} = \frac{\lambda f}{(1-\lambda\mu)A} + \left[\frac{\lambda f' + \lambda\delta - \lambda\mu}{1-\lambda\mu}\right]\frac{dk}{dA} > 0,$$

$$\frac{dy_a}{dA} = \frac{f}{A} + \left[f' + \frac{\lambda\mu f' + \delta - \mu}{1-\lambda\mu}\right]\frac{dk}{dA} > 0,$$

$$\frac{1}{\xi}\frac{dc}{dA} = \frac{1}{\lambda}\frac{ds}{dA} = \frac{dy_a}{dA} > 0,$$

where we require $\delta_k + \mu < 1$ in judging the sign of a and y_a. The effects of technological change are similar to the effects caused by changes in μ.

Effects of change in the propensity to hold money

We now examine effects of change in the propensity to hold money, ε_0. Taking derivatives of (3.3.11) with respect to ε_0 yields

$$-\frac{d\Omega}{dk}\frac{dk}{d\varepsilon_0} = -\frac{(\xi_0+\lambda_0)\mu\varepsilon^2}{\varepsilon_0^2}\frac{\lambda f - (1-\lambda\delta)k}{1-\lambda\mu} - (r+\mu-\varepsilon\mu)\frac{(f+\delta k)\lambda^2}{(1-\lambda\mu)\lambda_0}$$
$$-\frac{\mu\lambda^2 m}{(1-\lambda\mu)\lambda_0} - (f+\delta k)\frac{(\xi_0+\lambda_0)\varepsilon^2}{\varepsilon_0^2} < 0. \qquad (3.3.19)$$

A rise in ε_0 reduces the per capita physical wealth. As capital intensity falls, the rate of interest rises and per capita output level and wage rate rise. By equations (3.3.10), (3.3.4), and (3.3.5), we calculate

$$\frac{dm}{d\varepsilon_0} = -\frac{(f+\delta k)\lambda^2}{(1-\lambda\mu)\lambda_0} - \frac{\mu\lambda^2 m}{(1-\lambda\mu)\lambda_0} + \left[\frac{\lambda f' - (1-\lambda\delta)}{1-\lambda\mu}\right]\frac{dk}{d\varepsilon_0},$$

$$\frac{da}{d\varepsilon_0} = \frac{dk}{d\varepsilon_0} + \frac{dm}{d\varepsilon_0} = -\frac{(f+\delta k)\lambda^2}{(1-\lambda\mu)\lambda_0} - \frac{\mu\lambda^2 m}{(1-\lambda\mu)\lambda_0}$$
$$+\left[\frac{\lambda f' + \lambda\delta - \lambda\mu}{1-\lambda\mu}\right]\frac{dk}{d\varepsilon_0} < 0,$$

$$\frac{dy_a}{d\varepsilon_0} = -\frac{(f+\delta k)\mu\lambda^2}{(1-\lambda\mu)\lambda_0} - \frac{\mu^2\lambda^2 m}{(1-\lambda\mu)\lambda_0} + \left(\frac{f' + \delta - \mu}{1-\lambda\mu}\right)\frac{dk}{d\varepsilon_0} < 0,$$

$$\frac{dc}{d\varepsilon_0} = \xi\frac{d\bar{y}}{d\varepsilon_0} - \frac{\xi^2 y_a}{\xi_0} < 0, \quad \frac{ds}{d\varepsilon_0} = \lambda\frac{dy_a}{d\varepsilon_0} - \frac{\lambda^2 y_a}{\lambda_0} < 0.$$

The effects on the real money holdings are ambiguous. As the propensity to hold money is increased, the total wealth, consumption level and savings are all reduced.

Effects of change in the propensity to own wealth

We now examine effects of change in the propensity to own wealth, λ_0. Taking derivatives of (3.3.11) with respect to ε_0 yields

$$-\frac{d\Omega}{dk}\frac{dk}{d\lambda_0} = \left[\varepsilon + \frac{r+\mu-\varepsilon\mu}{1-\lambda\mu}(\varepsilon_0+\xi_0)\rho\right]\rho\mu m + \left[\frac{(f+\delta k)(\varepsilon_0+\xi_0)}{\lambda f-(1-\lambda\delta)k} - \frac{1}{\rho}\right](f+\delta k)\rho^2\varepsilon. \tag{3.3.20}$$

The impact of a rise in the propensity to save is ambiguous since the sign of

$$\frac{(f+\delta k)(\varepsilon_0+\xi_0)}{\lambda f-(1-\lambda\delta)k} - \frac{1}{\rho} = \frac{(\varepsilon_0+\xi_0)(f+\delta k+k)-[\lambda f-(1-\lambda\delta)k]\lambda_0}{[\lambda f-(1-\lambda\delta)k]\rho}$$

is ambiguous. Different from the model without money in which a rise in the propensity to save,[18] a rise in the propensity to save increases the per capita wealth in the model with money. As the impact on capital intensity is ambiguous, the effects on the rate of interest, per capita output level and wage rate are also ambiguous. We also calculate

$$\frac{dm}{d\lambda_0} = \frac{\lambda f'-(1-\lambda\delta)}{1-\lambda\mu}\frac{dk}{d\lambda_0} + (f+\delta k+\mu m)\frac{(\varepsilon_0+\xi_0)\lambda^2}{(1-\lambda\mu)\lambda_0^2}.$$

As the effects on the real money holdings and other variables are ambiguous, we will not represent the results.

3.4 A growth economy with two types of households

This section examines the same issues as in Section 2.3 but with the MIU approach. The production aspects of the economic system under consideration are similar to the OSG model in Section 3.3. We first define the model and then analyze its dynamic properties.

3.4.1 The MIUF-OSG growth model of heterogeneous households

Let there be 2 groups of consumers, indexed by j, in the economy. Each type of consumers has a fixed number of the population, denoted by N_j. The aggregated labor force N is given by

$$N = z_1N_1 + z_2N_2,$$

where z_j are the level of human capital of group j, $j = 1, 2$. We assume z_j to be fixed. We now study behavior of producers. Let $K(t)$ denote the capital stock at time t and N the flow of labor services used at t for production. We use the

conventional production function to describe a relationship between inputs and output. The rate of interest and wage rate are determined by markets. Hence, for any individual firm r and w_j are given at each point of time. The production sector chooses the two variables, K and N, to maximize its profit. Denote $\delta_k (0 \leq \delta_k < 1)$ the depreciation rate of physical capital. The marginal conditions are given by

$$r.(t) + \delta_k = f'(k(t)),$$
$$w_j(t) = z_j w(t), \quad w(t) \equiv f(k(t)) - k(t)f'(k(t)). \tag{3.4.1}$$

The ratio of wage rates is equal to the ratio of the human capital levels between the two groups.

The government levies no taxes. Money is introduced by assuming that a central bank distributes at no cost to the population a per capita amount of fiat money $M(t) > 0$. The scheme according to which the money stock evolves over time is deterministic and known to all agents. With μ being the constant net growth rate of the money stock, $M(t)$ evolves over time according to

$$\dot{M}(t) = \mu M(t), \quad \mu > 0.$$

The government expenditure in real terms per capita, $g(t)$, is given by

$$g(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t), \tag{3.4.2}$$

where $P(t)$ is the price of the output good and $m(t)$ stand for the real value of money per capita measured in units of the output good. Household j receives $\mu_j m(t)$ units of paper money from the government through a "helicopter drop" to each group, also considered to be independent of his money holdings.[19] The inflation rate, $\pi(t)$, is given by

$$\pi(t) \equiv \frac{\dot{P}(t)}{P(t)} = \mu - \frac{\dot{m}(t)}{m(t)}. \tag{3.4.3}$$

Household j's disposable income in real terms, $\hat{y}_j(t)$, is

$$\hat{y}_j(t) = r(t)k_j(t) + w_j(t) + \mu_j m(t) - \pi(t)m_j(t) + a_j(t), \tag{3.4.4}$$

where

$$a_j(t) \equiv k_j(t) + m_j(t).$$

The total money issued equals the total money received by the two groups

$$N_1 \mu_1 m(t) + N_2 \mu_2 m(t) = N_0 \mu m(t),$$

where $N_0(\equiv N_1 + N_2)$ is the population. We require $\mu_j \geq 0$ and

$$n_1\mu_1 + n_2\mu_2 = \mu,$$

where $n_j \equiv N_j/N_0$. We assume that at each point of time consumers' preferences over money, consumption and saving can be represented by the following utility function

$$U_j(t) = m_j^{\varepsilon_{0j}}(t)c_j^{\xi_{0j}}(t)s_j^{\lambda_{0j}}(t), \quad \varepsilon_{0j}, \xi_{0j}, \lambda_{0j} > 0,$$

where ε_{0j} is called household j's propensity to hold money, ξ_{0j} propensity to consume, and λ_{0j} propensity to own wealth. The budget constraint is given by

$$(1 + r(t))m_j(t) + c_j(t) + s_j(t) = \hat{y}_j(t).$$

Insert equation (3.4.4) into the budget constraints

$$(r + \pi)m_j + c_j + s_j = y_{aj} \equiv (1 + r)k_j + w_j + \mu m. \tag{3.4.5}$$

Consumers' problem is to choose current money, consumption, and savings in such a way that utility levels are maximized. Maximizing $U_j(t)$ subject to the budget constraint (3.4.5) yields

$$(r + \pi)m_j = \varepsilon_j y_{aj}, \quad c_j = \xi_j y_{aj}, \quad s_j = \lambda_j y_{aj}, \tag{3.4.6}$$

where

$$\varepsilon_j \equiv \rho_j\varepsilon_{0j}, \quad \xi_j \equiv \rho_j\xi_{0j}, \quad \lambda_j \equiv \rho_j\lambda_{0j}, \quad \rho_j \equiv \frac{1}{\varepsilon_{0j} + \xi_{0j} + \lambda_{0j}}.$$

The real wealth changes as follows

$$\dot{a}_j(t) = s_j(t) - a_j(t). \tag{3.4.7}$$

The total demand for money is equal to the total supply

$$N_0 m(t) = m_1(t)N_1 + m_2(t)N_2. \tag{3.4.8}$$

The assumption of full employment of capital is given by

$$K(t) = k_1(t)N_1 + k_2(t)N_2,$$

that is

$$k(t) = n_1k_1(t) + n_2k_2(t), \tag{3.4.9}$$

where $n_j \equiv N_j/N$. This equation expresses $k(t)$ as a unique function of $k_1(t)$ and $k_2(t)$. We have thus built the model.

3.4.2 The dynamics and equilibrium

First, we need to find equations which describe motion of the dynamic system. First, multiplying equation $(r+\pi)m_j = \varepsilon_j y_{aj}$ in (3.4.6) by N_j and then adding up the resulted two equations, we obtain

$$r+\pi = \frac{\varepsilon_1 N_1 y_{a1} + \varepsilon_2 N_2 y_{a2}}{m_1 N_1 + m_2 N_2}.$$

Substituting equation (3.4.8) into the above equation yields

$$\pi = \frac{\bar{\varepsilon}_1 y_{a1} + \bar{\varepsilon}_2 y_{a2}}{m} - r. \tag{3.4.10}$$

where $\bar{\varepsilon}_j \equiv \varepsilon_j N_j / N_0$. Insert equation (3.4.3) in equation (3.4.10)

$$\dot{m} = \Omega_m(k_1, k_2, m) \equiv (\mu + r)m - \bar{\varepsilon}_1 y_{a1} - \bar{\varepsilon}_2 y_{a2}. \tag{3.4.11}$$

As k is a unique function of k_1 and k_2 from equations (3.4.1) we see that r and w_j can also be expressed as unique functions of k_1 and k_2. From the definition of y_{aj}, y_{aj} are functions of k_1, k_2, and m. From $(r+\pi)m_j = \varepsilon_j y_{aj}$ and equation (3.4.7), we have

$$m_j = \Omega_j(k_1, k_2, m) \equiv \varepsilon_j y_{aj} m Y, \tag{3.4.12}$$

where

$$Y \equiv \frac{1}{\bar{\varepsilon}_1 y_{a1} + \bar{\varepsilon}_2 y_{a2}}.$$

Substituting $s_j = \lambda_j y_{aj}$ and $a_j = k_j + m_j$ into equations (3.4.13), we have

$$\dot{k}_j = \lambda_j R k_j + \lambda_j w_j + \lambda_j \mu m - k_j - \Omega_j - \dot{m}_j. \tag{3.4.13}$$

where $R \equiv 1 + r$. Taking derivatives of equations (3.4.12) with respect to t yields

$$\frac{\dot{m}_j}{m_j} = \frac{\Omega_m}{m} + \frac{\dot{y}_{aj}}{y_{aj}} - Y(\bar{\varepsilon}_1 \dot{y}_{a1} + \bar{\varepsilon}_2 \dot{y}_{a2}), \tag{3.4.14}$$

where

$$\dot{y}_{aj} = (n_1 \dot{k}_1 + n_2 \dot{k}_2)\tilde{k}_j + R\dot{k}_j + \mu\Omega_m,$$

in which we use equations (3.4.1) and (3.4.9) and

$$\tilde{k}_j \equiv (k_j - z_j k) f''.$$

Substituting equations (3.4.14) into equations (3.4.13) yields

$$\begin{aligned}\dot{k}_1 &= R_1 - (y_{a2}\dot{y}_{a1} - y_{a1}\dot{y}_{a2})\bar{\varepsilon}_2 Y_1,\\ \dot{k}_2 &= R_2 + (y_{a2}\dot{y}_{a1} - y_{a1}\dot{y}_{a2})\bar{\varepsilon}_1 Y_2,\end{aligned} \tag{3.4.15}$$

where $Y_j \equiv m_1 Y / y_{a1}$ and

$$R_j \equiv (\lambda_j R - 1)k_j + \lambda_j w_j + \lambda_j \mu m - \Omega_j - \frac{m_j \Omega_m}{m}.$$

We calculate

$$y_{a2}\dot{y}_{a1} - y_{a1}\dot{y}_{a2} = \tilde{R}_1 \dot{k}_1 + \tilde{R}_2 \dot{k}_2 + \tilde{R}_0, \tag{3.4.16}$$

where

$$\begin{aligned}\tilde{k} &\equiv Rk_2\tilde{k}_1 + w_2\tilde{k}_1 + \mu m\tilde{k}_1 - Rk_1\tilde{k}_2 - w_1\tilde{k}_2 - \mu m\tilde{k}_2,\\ \tilde{R}_1 &\equiv n_1\tilde{k} + RRk_2 + Rw_2 + R\mu m,\\ \tilde{R}_2 &\equiv n_2\tilde{k} - RRk_1\dot{k}_2 - Rw_1 - R\mu m,\\ \tilde{R}_0 &\equiv Rk_2\mu\Omega_m + w_2\mu\Omega_m - \mu Rk_1\Omega_m - \mu w_1\Omega_m.\end{aligned}$$

It should be noted that $\tilde{k}$ and $\tilde{R}_q$ are functions of k_1, k_2, and m. From equations (3.4.15) and (3.4.16), we solve

$$\begin{aligned}\dot{k}_1 &= \frac{R_1 - \bar{\varepsilon}_2 Y_1 \tilde{R}_0 - \bar{\varepsilon}_2 Y_1 \tilde{R}_2 \Omega}{1 + \bar{\varepsilon}_2 Y_1 \tilde{R}_1},\\ \dot{k}_2 &= \Omega(k_1, k_2, m)\\ &\equiv \left[R_2 + \bar{\varepsilon}_1 Y_2 \tilde{R}_0 + \frac{\left(R_1 - \bar{\varepsilon}_2 Y_1 \tilde{R}_0\right)\bar{\varepsilon}_1 Y_2 \tilde{R}_1}{1 + \bar{\varepsilon}_2 Y_1 \tilde{R}_1}\right]\left(1 - \frac{\bar{\varepsilon}_1 Y_2 \tilde{R}_2}{1 + \bar{\varepsilon}_2 Y_1 \tilde{R}_1}\right)^{-1}.\end{aligned} \tag{3.4.17}$$

Lemma 3.4.1

The motion of the economic system is described by three differential equations, (3.4.11) and (3.4.17) with $k_1(t), k_2(t)$, and $m(t)$ as variables. All the other variables are determined as functions of $k_1(t), k_2(t)$, and $m(t)$ by the following procedure: $k(t)$ by (3.4.9) $\rightarrow m_j(t), j = 1, 2$, by (3.4.12) $\rightarrow a_j(t) = k_j(t) + m_j(t) \rightarrow w_j(t)$ and $r(t)$ by (3.4.1) $\rightarrow K(t) = k(t)N \rightarrow f(t) = f(k(t)) \rightarrow F(t) = f(t)N \rightarrow \pi(t)$ by (3.4.10) $\rightarrow y_{aj}(t)$ by (3.4.5) $\rightarrow c_j(t)$ and $s_j(t)$ by (3.4.6).

As the expressions of the dynamics are complicated, we are only concerned with steady state. By equations (3.4.11) and (3.4.13), at steady state we have

$$\begin{aligned}&\mu m-\bar{\varepsilon}_1 y_{a1}-\bar{\varepsilon}_2 y_{a2}=0,\\&\lambda_j R k_j+\lambda_j w_j+\lambda_j \mu m-k_j-\varepsilon_j y_{aj} mY=0,\quad j=1,2.\end{aligned}\tag{3.4.18}$$

From the first equation in (3.4.18), we have $\mu mY=1$. Also from the first equation in (3.4.18) and $\bar{y}_j=Rk_j+w_j+\mu m$, we solve

$$m=\frac{(\bar{\varepsilon}_1 k_1+\bar{\varepsilon}_2 k_2)R+\bar{\varepsilon}w}{(1-\bar{\varepsilon})\mu},\tag{3.4.19}$$

where we use $w_j=z_j w$ and $\bar{\varepsilon}\equiv\bar{\varepsilon}_1+\bar{\varepsilon}_2$. Substituting $\mu mY=1$ and equation (3.4.19) into the last two equations in (3.4.18), we have

$$\begin{aligned}&\left(R+\frac{\bar{\varepsilon}_1 R}{1-\bar{\varepsilon}}-\bar{\lambda}_1\right)k_1+\left(z_1+\frac{\bar{\varepsilon}}{1-\bar{\varepsilon}}\right)w+\frac{(k-n_1k_1)\bar{\varepsilon}_2 R}{(1-\bar{\varepsilon})n_2}=0,\\&\left(R+\frac{\bar{\varepsilon}_2 R}{1-\bar{\varepsilon}}-\bar{\lambda}_2\right)\left(\frac{k-n_1k_1}{n_2}\right)+\left(z_2+\frac{\bar{\varepsilon}}{1-\bar{\varepsilon}}\right)w+\frac{\bar{\varepsilon}_1 k_1 R}{1-\bar{\varepsilon}}=0,\end{aligned}\tag{3.4.20}$$

where we use equation (3.4.9) and

$$\bar{\lambda}_j\equiv\left(\lambda_j-\frac{\varepsilon_j}{\mu}\right)^{-1}.$$

From the first equation in (3.4.20), we obtain

$$k_1=-n_2\left[\left(z_1+\frac{\bar{\varepsilon}}{1-\bar{\varepsilon}}\right)w+\frac{\bar{\varepsilon}_2 Rk}{(1-\bar{\varepsilon})n_2}\right]\left(\frac{n_2-n_2\bar{\varepsilon}_2-n_1\bar{\varepsilon}_2}{1-\bar{\varepsilon}}R-n_2\bar{\lambda}_1\right)^{-1}.\tag{3.4.21}$$

Erasing k_1 from equations (3.4.20), we obtain

$$\Omega(k)\equiv\left[(1-\bar{\varepsilon}_1)R\tilde{R}-(1-\bar{\varepsilon})\bar{\lambda}_2\tilde{R}-\bar{\varepsilon}_2 R\right]k+\left(\bar{z}_2\tilde{R}-\bar{z}_1\right)w=0,\tag{3.4.22}$$

where $\bar{z}_j\equiv\left[(1-\bar{\varepsilon})z_j+\bar{\varepsilon}\right]n_2$ and

$$\tilde{R}\equiv-\left(\frac{n_2-\bar{\varepsilon}_2 n_2-n_1\bar{\varepsilon}_2}{1-\bar{\varepsilon}}R-\bar{\lambda}_1 n_2\right)\left(\frac{n_1-n_1\bar{\varepsilon}_1-n_2\bar{\varepsilon}_1}{1-\bar{\varepsilon}}R+n_1\bar{\lambda}_2\right)^{-1}.$$

The equation, $\Omega(k) = 0$, contains a single variable, k. As the expression is tedious, it is difficult to check whether the equation has meaningful solutions. If the equation has a meaningful solution, then all the other variables are determined by the following procedure: $f = f(k) \to F = fN \to w_j$, $j = 1, 2$, and r by (3.4.1) $\to k_1$ by (3.4.21) $\to k_2$ by (3.4.9) $\to m$ by (3.4.19) $\to m_j$, by (3.4.12) $\to a_j = k_j + m_j \to K = kN \to \pi$ by (3.4.10) $\to y_{aj}$ by (3.4.5) $\to c_j$ and s_j by (3.4.6). As the expressions are tedious, it is difficult to get explicit conclusions in general if we do not specify the parameters. On the other hand, as the procedure for simulating the model is explicitly given, it is straightforward to simulate the model.

3.5 A portfolio balance MIUF-OSG growth model

This section develops a model similar to the model in Section 2.3, except that we introduce money to utility function. The only difference is that in this section money is held because it directly affects utility function, while in Section 2.3 money is held as a store of value.

3.5.1 The model with portfolio balance

The variables, $N(=1), K(t), F(K(t), N), y(t) = f(k(t)), k(t), \mu, \delta_k, M(t)$, and $m(t)$, $B(t)$, and $b(t)$ are defined as in Section 2.2. As before, we have

$$r + \delta_k = f'(k), \quad w = f(k) - kf'(k), \quad \dot{M} = \mu M, \quad \mu > 0,$$

$$\pi \equiv \frac{\dot{P}}{P} = \mu - \frac{\dot{m}}{m}. \tag{3.5.1}$$

Suppose that the economy contains three outside assets: fiat money, government bonds, and physical capital. The government budget constraint is expressed in real terms by

$$\dot{m}(t) + \dot{b}(t) = g - \tilde{\tau} + r(t)b(t) - \pi(t)(m(t) + b(t)), \tag{3.5.2}$$

where g and $\tilde{\tau}$ are respectively real government expenditure and real taxes and

$$\frac{\dot{M}(t)}{P(t)} = \dot{m}(t) + \pi(t)m(t), \quad \frac{\dot{B}(t)}{P(t)} = \dot{b}(t) + \pi(t)b(t). \tag{3.5.3}$$

The disposable income in real terms, $\hat{y}$, is

$$\hat{y} = rk + w + ib - \pi b - \pi m - \tilde{\tau} + a, \tag{3.5.4}$$

where $i(t)$ the nominal interest rate and $a(t) \equiv k(t) + b(t) + m(t)$. We have

$$r(t) = i(t) - \pi(t). \tag{3.5.5}$$

We assume that at each point of time the consumer's utility function for holding money, consuming goods and saving is be represented by the following utility function

$$U(t) = m^{\varepsilon_0}(t)c^{\xi_0}(t)s^{\lambda_0}(t), \quad \varepsilon_0, \xi_0, \lambda_0 > 0.$$

The budget constraint is given by

$$(1 + r(t))m(t) + c(t) + s(t) = \hat{y}(t).$$

Insert equation (3.5.4) into the budget constraint

$$(r + \pi)m + c + s = y_a \equiv (1 + r)k + w + (1 + r)b - \tilde{\tau}. \tag{3.5.6}$$

Consumers' problem is to choose current money, consumption, and savings in such a way that utility levels are maximized. Maximizing $U(t)$ subject to the budget constraint (3.5.6) yields

$$(r + \pi)m = \varepsilon y_a, \quad c = \xi y_a, \quad s = \lambda y_a, \tag{3.5.7}$$

where

$$\varepsilon \equiv \rho\varepsilon_0, \quad \xi \equiv \rho\xi_0, \quad \lambda \equiv \rho\lambda_0, \quad \rho \equiv \frac{1}{\varepsilon_0 + \xi_0 + \lambda_0}.$$

The real wealth changes as follows

$$\dot{a}(t) = s(t) - a(t). \tag{3.5.8}$$

We have thus built the model.

3.5.2 The dynamics and steady state

First, we need to find equations which describe motion of the dynamic system. From $\pi = \mu - \dot{m}/m$ and $(r + \pi)m = \varepsilon y_a$, we obtain

$$\dot{m} = \Omega_m(m, k, b) \equiv (r + \mu)m - (1 + r)\varepsilon k - \varepsilon w - (1 + r)\varepsilon b + \varepsilon\tilde{\tau}, \tag{3.5.9}$$

where we use equation (3.5.6). As r and w are functions of k, the right hand side of equation (3.5.9) is a function of k, m, and b. From equation (3.5.2), we get

$$\dot{b} = \Omega_b(m, k, b) \equiv \tilde{g} + rb - (m + b)\mu + \frac{b\Omega_m(m, k, b)}{m}, \tag{3.5.10}$$

where we use $\tilde{g} \equiv g - \tilde{\tau}$ and $\pi = \mu - \dot{m}/m$. We see that the right hand side of equation (3.5.10) is also a function of $k(t)$, $m(t)$, and $b(t)$.

Substituting $s = \lambda\bar{y}$ and the definition of a and equation (3.5.6) into equation (3.5.8), we have

$$\dot{k} = \bar{r}k + \lambda w + \bar{r}b - \lambda\tilde{\tau} - m - \Omega_m(m, k, b) - \Omega_b(m, k, b), \tag{3.5.11}$$

where $\bar{r}(t) \equiv \lambda + \lambda r(t) - 1$. We see that $\dot{k}(t)$ is a function of $k(t), m(t)$, and $b(t)$. The dynamics consist of three differential equations, (3.5.9)–(3.5.11), with three endogenous variables, $k(t)$, $m(t)$ and $b(t)$.

We now examine dynamic properties of the differential equations. From equations (3.5.9)–(3.5.11), a steady state is determined by

$$\begin{aligned}
&\left(\frac{r+\mu}{\varepsilon}\right)m - (1+r)k - w - (1+r)b + \tilde{\tau} = 0,\\
&\tilde{g} + rb - (m+b)\mu = 0,\\
&\bar{r}k + \lambda w + \bar{r}b - \lambda\tilde{\tau} - m = 0.
\end{aligned} \tag{3.5.12}$$

From the last two equations of (3.5.12), we solve m and b as functions of k

$$\begin{aligned}
b(k) &= \frac{\mu\bar{r}k + \mu\lambda w - \mu\lambda\tilde{\tau} - \tilde{g}}{r - \mu\lambda - \mu\lambda r},\\
m(k) &= \lambda w + \frac{(kr - \mu k + \mu\lambda w - \mu\lambda\tilde{\tau} - \tilde{g})\bar{r}}{r - \mu\lambda - \mu\lambda r} - \lambda\tilde{\tau}.
\end{aligned} \tag{3.5.13}$$

Substituting equations (3.5.13) into the first equation in (3.5.12) yields

$$\Omega(k) \equiv \left(\frac{r+\mu}{\varepsilon}\right)m(k) - (1+r)k - w - (1+r)b(k) + \tilde{\tau} = 0.$$

From this equation, we can determine the equilibrium value of k. As the expressions are tedious, it is difficult to interpret conditions under which the system has steady states.

For illustration, assume that the system has a steady state. We now examine effects of change in the inflation policy on the steady state. Taking derivatives of equations (3.5.12), we have

$$\begin{pmatrix} f_0 & \frac{r+\mu}{\varepsilon} & -1-r \\ f''b & -\mu & r-\mu \\ \bar{r} + \lambda b f'' & -1 & \bar{r} \end{pmatrix} \begin{pmatrix} \frac{dk}{d\mu} \\ \frac{dm}{d\mu} \\ \frac{db}{d\mu} \end{pmatrix} = \begin{pmatrix} -\frac{m}{\varepsilon} \\ m+b \\ 0 \end{pmatrix}, \tag{3.5.14}$$

where

$$f_0 \equiv \frac{mf''}{\varepsilon} - f''k - 1 - r - kf'' - bf''$$

From (3.5.14) we can determine the effects on k, m, and b. From this result, it is straightforward to calculate the effects on all the other variables.

3.6 Growth with capital accumulation and education

Adam Smith held that the large gains in the productivity of labor have their origins in the large part of the skill, dexterity, and judgment, which are consequences of the division of labor. The opportunities and incentives to which workers respond in their investment in human capital are not seriously considered by classical economists such as Smith, Ricardo, and Marx. The omission was perhaps not so much misleading as it might appear, given the role of innovation and education in economic development when the classical economists were constructing their theories.

One of the first seminal attempts to render technical progress endogenous in growth models was initiated by Arrow in 1962. He emphasized one aspect of knowledge accumulation – learning by doing. In 1965 Uzawa introduced a sector specifying in creating knowledge into growth theory. The knowledge sector utilizes labor and the existing stock of knowledge to produce new knowledge, which enhances productivity of the production sector. Another approach is taken by, for instance, Kennedy in 1964, Weisäcker in 1966 and Samuelson in 1965, who took account of the assumption of "inducement through the factor prices." In 1981 Schultz emphasized the incentive effects of policy on investment in human capital.[20] There are many other studies on endogenous technical progresses.[21] But on the whole theoretical economists had been relatively silent on the topic from the end of the 70s until the publication of Romer's 1986 paper.[22] Since then there has been an even increasing number of publications on the literature. In Romer's approach, knowledge is taken as an input in the production function and competitive equilibrium is rendered consistent with increasing aggregate returns owing to externalities. It is assumed that knowledge displays increasing marginal productivity but new knowledge is produced by investment in research technology, which exhibits diminishing returns. Various other issues related to innovation, diffusion of technology and behavior of economic agents under various institutions have been discussed in the literature. There are also many other models emphasizing different aspects, such as education, trade, R&D policies, entrepreneurship, division of labor, learning through trading, brain drain, economic geography, of dynamic interactions among economic structure, development and knowledge.[23] The purpose of this section is to introduce a growth model with endogenous human capital accumulation. The model treats Arrow's learning by doing and Uzawa's education as two main sources of human capital accumulation. Another important issue we will address in this section is path-dependent economic development. An economy's long-term prosperity may depend on initial conditions is nowadays a familiar idea in the growth literature. Many models capture different aspects of this kind of phenomena in formal models.[24]

It is evident that as money is important in the neoclassical growth theory and the Uzawa model with education and capital deals with interactions of two important growth factors, it may provide some new insights to introduce money into the Uzawa model.[25] This section introduces money into the OSG model with endogenous human capital proposed by Zhang (2006a: Section 3.2).

3.6.1 The monetary growth model with endogenous human capital

The economy has one production sector and one education sector. The latter is called the university. We assume a homogenous and fixed national labor force, N. The labor force is distributed between economic activities, teaching, and studying. We select commodity to serve as numeraire, with all the other prices being measured relative to its price. We assume that wage rates are identical among all professions. We introduce

$F(t)$: output level of the production sector at time t;
$K(t)$: level of capital stocks of the economy;
$H(t)$: level of human capital of the population;
$N_i(t)$ and $K_i(t)$: labor force and capital stocks employed by the production sector;
$N_v(t)$ and N_e: number of teachers and number of students;
$K_e(t)$: capital stocks employed by the university; and
$w(t)$ and $r(t)$: wage rate and rate of interest.

We first model production and consumption. We assume that production is to combine qualified labor force, $H^{\vartheta}(t)N_i(t)$, and physical capital, $K_i(t)$. Most aspects of our model are the same as the OSG model. The production process is described by

$$F(t) = AK_i^{\alpha}(t)(H^{\vartheta}(t)N_i(t))^{\beta}, \quad A, \alpha, \beta > 0, \alpha + \beta = 1.$$

Let τ stand for the (fixed) tax rate on the product level. The marginal conditions are given by

$$r(t) + \delta_k = \overline{\tau}\alpha A k_i^{-\beta}(t), \quad w(t) = \overline{\tau}\beta A H^{\vartheta}(t) k_i^{\alpha}(t), \tag{3.6.1}$$

where $k_i \equiv K_i/H^{\vartheta}N_i$ and $\overline{\tau} \equiv 1 - \tau$.

With μ being the constant net growth rate of the money stock, $M(t)$ evolves over time according $\dot{M}(t) = \mu M(t)$, $\mu > 0$. The government income in real terms per capita from issuing money is given by

$$\frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t). \tag{3.6.2}$$

We assume that this income is spent on education.

We denote per capita wealth by $k(t)$, where $k(t) \equiv K(t)/N$. The disposable income in real terms, $\hat{y}$, is

$$\hat{y} = rk + w - \pi m + a, \tag{3.6.3}$$

where $a(t) \equiv k(t) + m(t)$. The consumer's utility function is given by $U = m^{\varepsilon_0} c^{\xi_0} s^{\lambda_0}$. Similar to equation (3.3.4), we have the budget constraint

$$(\pi + r)m + c + s = \overline{y} = (1 + r)k + w. \tag{3.6.4}$$

Maximizing $U(t)$ subject to the budget constraint yields

$$(r+\pi)m = \varepsilon\bar{y}, \quad c = \xi\bar{y}, \quad s = \lambda\bar{y}, \tag{3.6.5}$$

where

$$\varepsilon \equiv \rho\varepsilon_0, \quad \xi \equiv \rho\xi_0, \quad \lambda \equiv \rho\lambda_0, \quad \rho \equiv \frac{1}{\varepsilon_0 + \xi_0 + \lambda_0}.$$

The real wealth changes as follows

$$\dot{a}(t) = s(t) - a(t). \tag{3.6.6}$$

We now study the behavior of the university. We assume that there are two sources of improving human capital, through education and learning by producing. Arrow first introduced learning by doing into growth theory; Uzawa took account of trade offs between investment in education and capital accumulation.[26] We propose that human capital dynamics is given by[27]

$$\dot{H} = \frac{\upsilon_e K_e^{\alpha_e} \left(H^{\vartheta} N_v\right)^{\beta_v} \left(H^{\vartheta} N_e\right)^{\beta_e}}{N} + \frac{\upsilon_i F}{NH^{\upsilon}} - \delta_h H, \tag{3.6.7}$$

where $\delta_h(>0)$ is the depreciation rate of human capital, $\upsilon_e, \upsilon_i, \alpha_e, \beta_v$, and β_e are non-negative parameters. The above equation is a synthesis and generalization of Arrow's and Uzawa's ideas about human capital accumulation. The term, $\upsilon_e K_e^{\alpha_e} \left(H^{\vartheta} N_v\right)^{\beta_v} \left(H^{\vartheta} N_e\right)^{\beta_e}$, describes the contribution to human capital improvement through education. Human capital tends to increase with an increase in the number of students. The term divided by N measures the contribution per capita. We take account of learning by doing effects in human capital accumulation by the term $\upsilon_i F/H^{\upsilon}$. This term implies that contribution of the production sector to human capital improvement is positively related to its production scale F and is dependent on the level of human capital. The term H^{υ} takes account of returns to scale effects in human capital accumulation. The case of $\upsilon > (<)0$ implies that as human capital is increased it is more difficult (easier) to further improve the level of human capital.

We assume that the students and teachers are paid by government's tax income at the same wage rate as the wage rate of workers. We assume that the economy has a fixed ratio of the population who are getting education in the university. The number of students is given by $N_e = n_e N$. Assume that the total tax income is used for paying the students, teachers and the capital stocks employed by the university. The government spends $w(t)N_e$ amount of money on students. Obviously, this assumption is strict. A way to relax this assumption without increasing analytical complex of the model is to assume that each individual spends a given amount of time on education. The time distribution is given by $T_i + T_e = T$, where T is the fixed available time, T_i is the work time and T_e is the time as a student.

The student gets free education but does not receive any wage. It can be seen that the conclusions will not be different in the two cases and the model is still analytically tractable. The budget for paying teachers and the capital stocks of the university is given by

$$w(t)N_v(t)+(r(t)+\delta_k)K_e(t)=G(t)\equiv\tau F(t)+\mu Nm(t)-w(t)N_e. \tag{3.6.8}$$

The university distributes its total resource $\tau F+\mu Nm-wN_e$ to the teachers $N_v(t)$ and the capital stocks $K_e(t)$ in such a way that the output of the university $\upsilon_e K_e^{\alpha_e}\left(H^{\vartheta}N_v\right)^{\beta_v}\left(H^{\vartheta}N_e\right)^{\beta_e}$ will be maximized. The university's problem is formed as follows

$$Max\ \upsilon_e K_e^{\alpha_e}\left(H^{\vartheta}N_v\right)^{\beta_v}\left(H^{\vartheta}N_e\right)^{\beta_e}$$

subject to equation (3.6.5). The solution is given by

$$K_e=\frac{\alpha_e G}{\alpha_0 r},\quad N_v=\frac{\beta_v G}{\alpha_0 w}, \tag{3.6.9}$$

where $\alpha_0\equiv\alpha_e+\beta_v$. We assume that labor force and capital stocks are fully employed

$$N_i+N_e+N_v=N,\quad K_i+K_e=K. \tag{3.6.10}$$

We have thus built the model.

3.6.2 The motion of money, wealth, and human capital

First, from $(r+\pi)m=\varepsilon\bar{y}$ in equations (3.6.5) and the definition of $\bar{y}$, we get

$$\dot{m}=(\mu+r)m-(1+r)\varepsilon k-\varepsilon w, \tag{3.6.11}$$

where we also use $\pi=\mu-\dot{m}/m$. We see that the right-hand side of equation (3.6.1) is a function of k, H, m, and k_i.

From $N_e=n_eN$ and equations (3.6.10) we get

$$N_i+N_v=(1-n_e)N.$$

From equations (3.6.1) and (3.6.9), we get

$$\begin{aligned}K_e&=\frac{\alpha_e\left(\tau AH^{\vartheta}k_i^{\alpha}N_i+\mu Nm-wN_e\right)}{\alpha\alpha_0\bar{\tau}Ak_i^{-\beta}},\\ N_v&=\frac{\beta_v\left(\tau AH^{\vartheta}k_i^{\alpha}N_i+\mu Nm-wN_e\right)}{\alpha_0\beta\bar{\tau}AH^{\vartheta}k_i^{\alpha}},\end{aligned} \tag{3.6.12}$$

where we use $F = AH^m k_i^\alpha N_i$. From $N_i + N_v = (1 - n_e)N$ and the second equation in (3.6.12), we solve

$$N_i = n_i - \frac{\bar{n}m}{H^\vartheta k_i^\alpha}, \quad N_v = n_v + \frac{\bar{n}m}{H^\vartheta k_i^\alpha}, \tag{3.6.13}$$

where

$$n_i \equiv \frac{(\alpha_0 + \beta_v)(1-\tau)\beta N_e}{\tau\beta_v + \alpha_0\bar{\tau}\beta}, \quad \bar{n} \equiv \frac{\mu\beta_v N}{(\tau\beta_v + \alpha_0\bar{\tau}\beta)A}, \quad n_v = \frac{(\tau - \bar{\tau}\beta)\beta_v N_e}{\tau\beta_v + \alpha_0\bar{\tau}\beta}.$$

We see that the labor distribution is determined as functions of H, m, and k_i. Substituting N_i in equations (3.6.13) into the first equation in (3.6.12) yields

$$K_e = \left(n_v H^\vartheta k_i + \alpha_e \bar{n} m k_i^\beta\right) \frac{\beta}{\alpha\beta_v}. \tag{3.6.14}$$

From $N_i = n_i - \bar{n}m/H^\vartheta k_i^\alpha$ and the definition of k_i, we have

$$K_i = n_i H^\vartheta k_i - \bar{n} k_i^\beta m. \tag{3.6.15}$$

From $K = kN$ and equations (3.6.14) and (3.6.15), we see that K, K_i, and K_e are functions of k, H, m, and k_i. Substituting equations (3.6.14) and (3.6.15) into $K_i + K_e = kN$ yields

$$k = \Lambda_0(k_i, H, m) \equiv \bar{n}_0 H^\vartheta k_i + \bar{n}_1 k_i^\beta m, \tag{3.6.16}$$

where

$$\bar{n}_0 \equiv \left(\frac{n_v\beta}{\alpha\beta_v} + n_i\right)\frac{1}{N}, \quad \bar{n}_1 \equiv \frac{(\alpha_e\beta - \alpha\beta_v)\bar{n}}{\alpha\beta_v N}.$$

The per capita physical wealth is a function of H, m, and k_i. Substituting equations (3.6.16) and (3.6.1) into equation (3.6.11) yields

$$\dot{m} = \Lambda_m(k_i, H, m) \equiv \left(\mu - \delta_k + \alpha\bar{\tau}Ak_i^{-\beta}\right)m - \left(1 - \delta_k + \alpha\bar{\tau}Ak_i^{-\beta}\right)\varepsilon\Lambda_0 - \bar{\tau}\varepsilon\beta AH^\vartheta k_i^\alpha. \tag{3.6.17}$$

From $F = AH^\vartheta k_i^\alpha N_i$, and equations (3.6.7), (3.6.14)–(3.6.16), we obtain

$$\dot{H} = \Lambda_H(k_i, H, m) \equiv \upsilon_e H^{\vartheta(\beta_v + \beta_e)}\left(n_v + \frac{\bar{n}m}{H^\vartheta k_i^\alpha}\right)^{\beta_v}\left(n_v H^\vartheta k_i^+ \alpha_e \bar{n} m k_i^\beta\right)^{\alpha_e} + \left(n_i - \frac{\bar{n}m}{H^\vartheta k_i^\alpha}\right)\frac{\upsilon_i A H^{\vartheta - \nu} k_i^\alpha}{N} - \delta_h H, \tag{3.6.18}$$

where

$$\overline{\upsilon}_e \equiv \frac{\upsilon_e N_e^{\beta_e} (\beta/\alpha\beta_v)^{\alpha_e}}{N}$$

We see that $\dot{m}$ and $\dot{H}$ are functions of $H(t), m(t)$, and $k_i(t)$. We now find out the motion of k_i.

From equation (3.6.6), $a = k + m$, and $s = \lambda\bar{y}$, we have

$$\dot{k} = \lambda\bar{y} - k - m - \dot{m}.$$

Inserting the definition of $\bar{y}$ and equation (3.6.17) in the above equation, we obtain

$$\dot{k} = \overline{\Lambda}k + \lambda\overline{\tau}\beta A H^{\vartheta} k_i^{\alpha} - m - \Lambda_m, \tag{3.6.19}$$

where we also use equations (3.6.1) and

$$\overline{\Lambda}(k_i) \equiv \lambda\overline{\tau}\alpha A k_i^{-\beta} - \delta, \quad \delta \equiv 1 - \lambda + \delta_k\lambda.$$

Taking derivatives of equation (3.6.16) with respect to t yields

$$\dot{k} = \left(\overline{n}_0 H^{\vartheta} + \beta\overline{n}_1 k_i^{-\alpha} m\right)\dot{k}_i + m\overline{n}_0 H^{\vartheta-1} k_i \Lambda_H + \overline{n}_1 k_i^{\beta} \Lambda_m,$$

where we also use equations (3.6.17) and (3.6.18). Substituting the above equation and equation (3.6.16) into equation (3.5.19), we have

$$\begin{aligned}\dot{k}_i &= \Lambda_{k_i}(k_i, H, m) \\ &\equiv \frac{\overline{\Lambda}\Lambda_0 + \lambda\overline{\tau}\beta A H^{\vartheta} k_i^{\alpha} - m - \left(1 + \overline{n}_1 k_i^{\beta}\right)\Lambda_m - m\overline{n}_0 H^{\vartheta-1} k_i \Lambda_H}{\overline{n}_0 H^{\vartheta} + \beta\overline{n}_1 k_i^{-\alpha} m}\end{aligned} \tag{3.6.20}$$

Lemma 3.6.1

The motion of $m(t), H(t)$, and $k_i(t)$ is determined by the three differential equations

$$\begin{aligned}\dot{m} &= \Lambda_m(k_i, H, m), \\ \dot{H} &= \Lambda_H(k_i, H, m), \\ \dot{k}_i &= \Lambda_{k_i}(k_i, H, m).\end{aligned} \tag{3.6.21}$$

For any positive $m(t), H(t)$, and $k_i(t)$, all the other variables are uniquely determined at any point of time by the following procedure: k by (3.6.16) $\rightarrow$ $K = kN \rightarrow K_i$ by (3.6.15) $\rightarrow K_e$ by (3.6.14) $\rightarrow N_i$ and N_v by (3.6.13) $\rightarrow r$ and w by (3.6.1) $\rightarrow \bar{y}$ by (3.6.4) $\rightarrow \pi, c$ and s by (3.6.5) $\rightarrow a = k + m \rightarrow F = AK_i^{\alpha}(H^m N_i)^{\beta}$.

The lemma is important in that it provides a procedure for computing the motion of the entire system. For any proper initial conditions, it is straightforward to

simulate the model with computer. As the expressions are tedious, we don't discuss under what conditions the dynamic system has meaningful solutions. We now study steady states.

At steady state we have $\pi = \mu$ and $a = \lambda\bar{y}$. From $a = \lambda\bar{y}$ and the definition of $\bar{y}$, we have

$$k = \frac{m - \bar{\tau}\beta A H^{\vartheta} k_i^{\alpha}}{\bar{\Lambda}(k_i)}. \tag{3.6.22}$$

where we use equations (3.6.1). From $(r+\pi)m = \varepsilon\bar{y}$, $\pi = \mu$ and $a = \lambda\bar{y}$, we have

$$(r+\mu)\lambda m = \varepsilon k + \varepsilon m.$$

Substituting equation (3.6.22) into the above equation, we solve

$$m = \frac{\varepsilon\bar{\tau}\beta A H^{\vartheta} k_i^{\alpha}}{(\lambda r + \lambda\mu - \varepsilon)\bar{\Lambda}(k_i) + \varepsilon}. \tag{3.6.23}$$

From equation (3.6.16) and (3.6.22), we have

$$m = \frac{\bar{n}_0 \bar{\Lambda} H^{\vartheta} k_i + \bar{\tau}\beta A H^{\vartheta} k_i^{\alpha}}{1 - \bar{n}_1 \bar{\Lambda} k_i^{\beta}}.$$

From this equation and equation (3.6.23), we solve

$$k_i^{2\beta} - a_1 k_i^{\beta} + a_2 = 0, \tag{3.6.24}$$

where we require $\bar{\Lambda}(k_i) \neq 0$ and

$$a_1 \equiv \left[\frac{\delta\bar{n}_0\lambda\alpha + (\lambda\alpha\bar{n}_0 + \beta)(\lambda\delta_k + \varepsilon - \lambda\mu)}{(\lambda\delta_k + \varepsilon - \lambda\mu)\delta\bar{n}_0 + (\bar{\tau}\beta A\bar{n}_1 + \bar{n}_0)\varepsilon}\right]\bar{\tau}A,$$

$$a_2 \equiv \frac{(\lambda\alpha\bar{n}_0 + \beta)\lambda\alpha\bar{\tau}^2 A^2}{(\lambda\delta_k + \varepsilon - \lambda\mu)\delta\bar{n}_0 + (\bar{\tau}\beta A\bar{n}_1 + \bar{n}_0)\varepsilon}.$$

From their definitions, we can see that $\bar{n}_0 > 0$ and $n_i > 0$. The numerator of a_2 is positive. It is not strict to require $\lambda\delta_k + \varepsilon > \lambda\mu$. As

$$\bar{\tau}\beta A\bar{n}_1 + \bar{n}_0 = \left\{\frac{\tau N_e}{N} + \left[(\alpha\alpha_0 + \alpha\beta_v - \beta)\frac{N_e}{N} + (\alpha_e\beta - \alpha\beta_v)\mu\right]\bar{\tau}\right\}\frac{\beta}{\alpha(\tau\beta_v + \alpha_0\bar{\tau}\beta)}.$$

we see that $\overline{\tau}\beta A\overline{n}_1 + \overline{n}_0$ is positive if $\alpha_v + 2\beta_v > \beta/\alpha$ and $(\alpha_e\beta - \alpha\beta_v)\mu$ are small. If $\lambda\delta_k + \varepsilon > \lambda\mu$ and $\overline{\tau}\beta A\overline{n}_1 + \overline{n}_0 > 0$, then we have $a_2 > 0$. If the conditions for $a_2 > 0$ are satisfied, then $a_1 > 0$. Equation (3.6.24) has a unique positive solution[28]

$$k_i = \left(\frac{a_1 + \sqrt{a_1^2 - 4a_2}}{2}\right)^{1/\beta}. \tag{3.6.25}$$

From equation (3.6.23), we have $m = m_0 H^m$ where

$$m_0 \equiv \frac{\varepsilon\overline{\tau}\beta A k_i^{\alpha}}{(\lambda r + \lambda\mu - \varepsilon)\overline{\Lambda}(k_i) + \varepsilon}.$$

is a constant (as k_i is already uniquely determined). Substituting $m = m_0 H^m$ into equation (3.6.18) at steady state, we have

$$\Omega(H) \equiv \overline{v}_{e0} H^{x_e} + \overline{v}_i H^{x_i} - \delta_h = 0, \tag{3.6.26}$$

where

$$\Omega_e(H) \equiv \overline{v}_{e0} H^{x_e}, \quad \Omega_i(H) \equiv \overline{v}_i H^{x_i},$$

$$x_e \equiv (\beta_v + \beta_e + \alpha_e)\vartheta - 1, \quad x_i \equiv \vartheta - \nu - 1,$$

$$\overline{v}_{e0} \equiv \overline{v}_e \left(n_v + \frac{\overline{n}m_0}{k_i^{\alpha}}\right)^{\beta_v} \left(n_v k_i^{+} \alpha_e \overline{n} m_0 k_i^{\beta}\right)^{\alpha_e}, \quad \overline{v}_i \equiv \left(n_i - \frac{\overline{n}m_0}{k_i^{\alpha}}\right)\frac{v_i A k_i^{\alpha}}{N}.$$

We see that the number of economic equilibrium points is equal to the number of solutions of the equation, $\Phi(H) = 0$ for $0 < H < \infty$. As shown in Figure 3.6.1, the equation may have none, one equilibrium point, or two equilibrium points. As shown in Figure 3.6.1a, if $x_e < 0$ and $x_i < 0$, the equation monotonically decreases in H and it passes the horizontal axis only once. Figure 3.6.1b depicts the case of $x_e > 0$ and $x_i > 0$, the function monotonically increases in H and it passes the horizontal axis only once. Figure 3.6.1c depicts the case of $x_e > 0$ and $x_i < 0$ ($x_e < 0$ and $x_i > 0$).

In Appendix A.3.1, we explicitly provide check the conditions in Figure 3.6.1. The following lemma shows that the properties of the dynamic system are determined by the two returns to scale parameters, x_e and x_i.

Lemma 3.6.2

(1) If $x_e < 0$ and $x_i < 0$, the system has a unique stable equilibrium; (2) If $x_e > 0$ and $x_i > 0$, the system has a unique unstable equilibrium; and (3) If $x_e > 0$ and $x_i < 0$ ($x_e < 0$ and $x_i > 0$), the system has either no equilibrium, one equilibrium or two equilibrium points. For a solution H, all the other variables are given by following procedure: $m = m_0 H^m \to k_i$ by (3.6.25) $\to k$ by (3.6.22) $\to K = kN \to K_i$ by

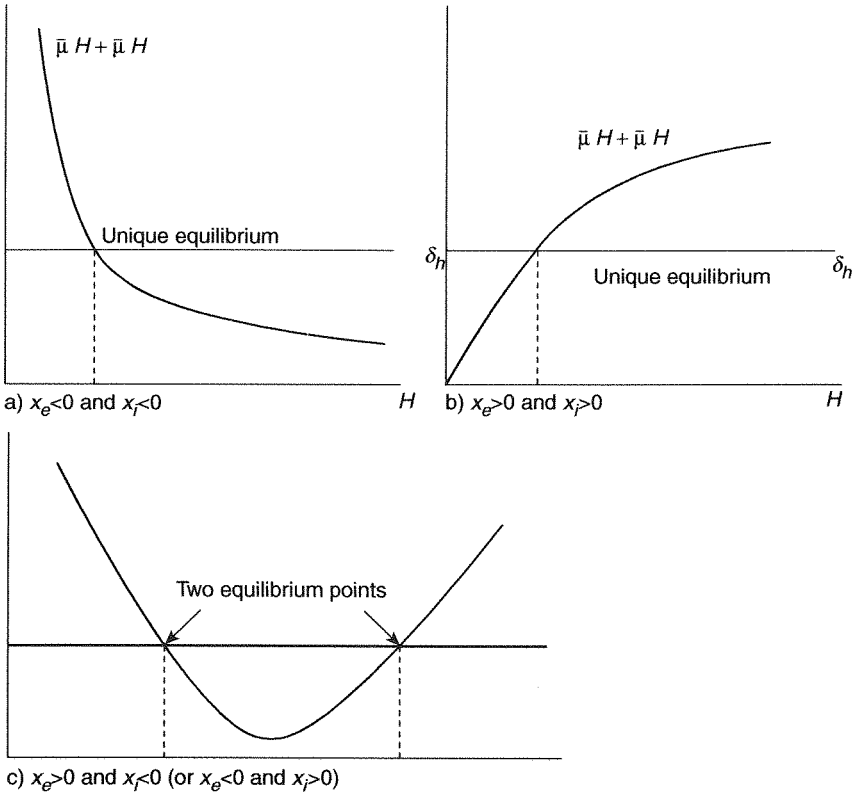


Figure 3.6.1 The two sectors exhibit different returns to scale effects.

(3.6.15) $\rightarrow K_e$ by (3.6.14) $\rightarrow N_i$ and N_v by (3.6.13) $\rightarrow r$ and w by (3.6.1) $\rightarrow \bar{y}$ by (3.6.4) $\rightarrow \pi, c$ and s by (3.6.5) $\rightarrow a = k + m \rightarrow F = AK_i^{\alpha}(H^m N_i)^{\beta}$.

Lemma 3.6.1 provides a procedure to determine the equilibrium values of all the variables in the system. For illustration, we simulate the motion of the dynamic system. We limit our examination to the case of nonmonetary economy.[29]

3.6.3 Simulate the nonmonetary economy

We let $\varepsilon_0 = 0$, and require $\xi + \lambda = 1$ where we omit subscript 0 for ξ and λ.[30] We specify the parameters as follows

$$\alpha = 0.35, N = 1, A = 2, N_e = 0.06, \tau = 0.08, \lambda = 0.7, \alpha_e = 0.7,$$
$$\beta_e = 0.7, \beta_v = 0.7, \nu_e = 1.8, \nu_i = 0.02, \delta_h = 0.08, \nu = 0.3, \vartheta = 0.8. \quad (3.6.27)$$

Under the above specifications, we have $x_e = 0.4$ and $x_i = -0.5$. The university exhibits increasing returns to scale and the production sector is characterized of

decreasing returns. The system has two equilibrium points

$$(k_1, H_1) = (17.858, 2.226), \quad (k_2, H_2) = (10.401, 19.509).$$

The one with lower levels of human capital and per capita wealth is stable; the other is unstable, also as analytically proved. Figure 3.6.2 depicts the vector field and the steady states of the dynamic system. As shown in Figure 3.6.2, an economy with low level of human capital, even if it was initially rich, tends to converge to the stable equilibrium with low standard of living and low level human capital. An economy with high level of human capital, even if it was initially poor, tends to experience sustained growth. This nonlinear dynamic system has path-dependent features. Here, we can see the significance of cultural value for education. Japanese, Korean, and Chinese-dominated economies like Singapore and Taiwan could have sustained economic growth irrespective of their initial poor conditions in the 1960s, mainly because of their cultural values on education, rather than due to high saving rates.[31] In the 1950s, no one could have foreseen rapid economic development of East Asia, because few economists recognized the significance of education in economic development and even fewer still knew the validity of rationalism in classical Confucianism for modern economies.[32] Economists failed to properly examine the cultural values of education in this region.

We simulate three paths of the economy with the initial values

$$(k_0, H_0) = (1.5, 10), \quad (k_0, H_0) = (115, 12), \quad (k_0, H_0) = (70, 23).$$

The paths with $(k_0, H_0) = (1.5, 10)$ and $(k_0, H_0) = (115, 12)$ converge to the low levels of human capital and per capita wealth. It is interesting to note that the path

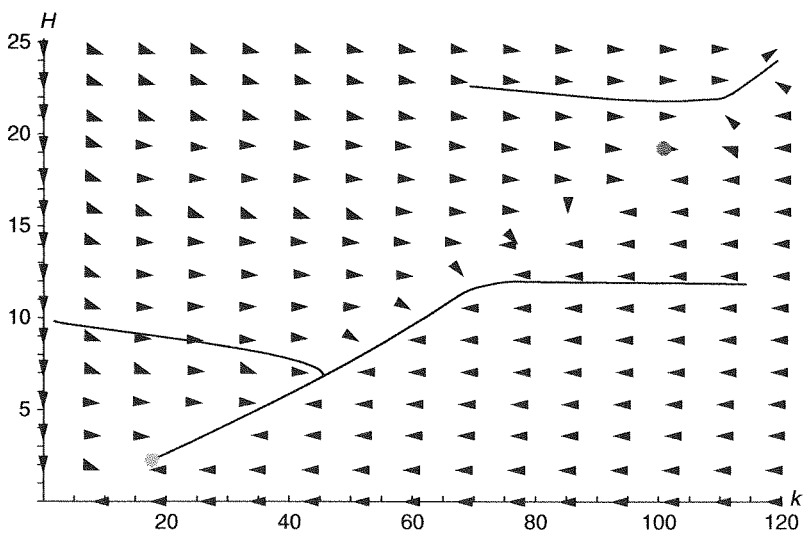


Figure 3.6.2 Path dependent economic evolution.

with $(k_0, H_0) = (115, 12)$ starts with high level of wealth. But its level of human capital is not improved over time. As decreasing returns dominate this path, its prosperity does not last long. The path with $(k_0, H_0) = (70, 23)$ will grow infinitely because the increasing return to scale dominates the economic evolution. Indeed, this kind of infinite growth will not happen in reality as our model neglects many other significant factors such as population change, negative externalities such as pollution, and limitations of natural resources.

The impact of the education policy

In our system, the tax income is totally spent on education. We may thus interpret increases in the tax rate as the promotion policy taken by the government. We now examine the impact of the tax rate on the economic structure. Intuitively, an increase in tax rate may either increase or decrease output because education costs resources and may have little impact on human capital improvement. We now show under what conditions an increase in the tax rate may promote the economy. We still specify the parameter values as in (3.6.27) except the tax rate, τ. We reduce the resource for education. Let us consider the case that the expenditure on education is reduced from 8 percent of the GDP to 7 percent.

Figure 3.6.3 shows the simulation results – the points with larger sizes is the new steady states and the other two points with smaller sizes are the old steady states. The two steady states shift as follows:

$$(k_1, H_1) : (17.858, 2.226) \Rightarrow (14.758, 1.718),$$

$$(k_2, H_2) : (101.401, 19.509) \Rightarrow (230.783, 53.417).$$

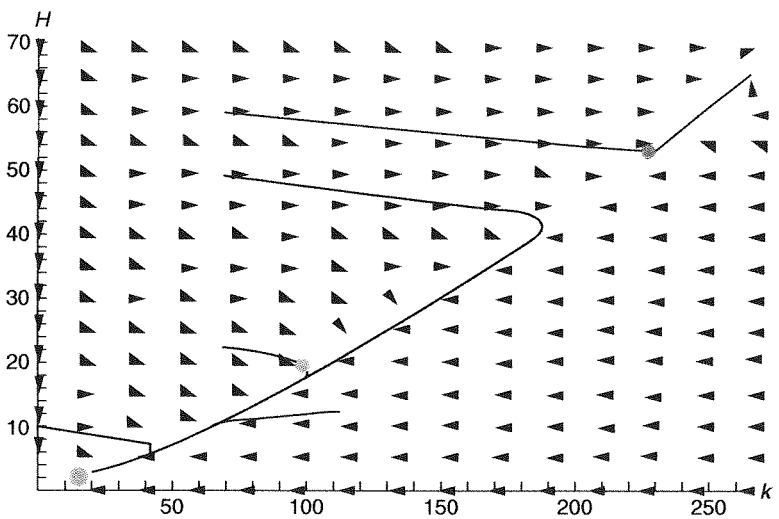


Figure 3.6.3 The path dependent-development as the education is discouraged.

We see that the new stable steady state has lower levels of human capital and per capita wealth; but the new unstable steady states have much higher levels of k and H. It seems promising with the new education policy because the new unstable steady state of higher k and H is much better than the old unstable one. Nevertheless, the economy with the discouraging policy has more chances to the traditional trap than to the economic miracle. For instance, if we start from the following three points as in the previous example in Figure 3.6.2:

$$(k_0, H_0) = (1.5, 10), \quad (k_0, H_0) = (115, 12), \quad (k_0, H_0) = (70, 23).$$

As demonstrated in Figure 3.6.3, all the paths with these initial conditions end up in the poverty trap. But in Figure 3.6.2, the path with $(k_0, H_0) = (70, 23)$ exhibits the economic miracle. This example shows that the discouraging policy deprives the society from development opportunity. The "chance" for development is less due to the new policy. As shown in Figure 3.6.3, for the economy to experience sustained growth, the economy must have a much higher initial level of human capital than in the case of $\tau = 0.08$. Hence, if the society reduces its investment in education, it will have much less opportunities to experience sustained economic growth, even though heavy investment in education will not guarantee sustainable development of the nonlinear system in certainty.

The impact of the propensity to save

To simulate the impact of change in the propensity to save, we still specify the parameters values as in (3.3.27), except the propensity to save λ. We increase the propensity to save. Let us consider the case that the propensity to save is increased from 0.7 to 0.73, that is, $\lambda : 0.7 \Rightarrow 0.73$. Figure 3.6.4 shows the simulation results –

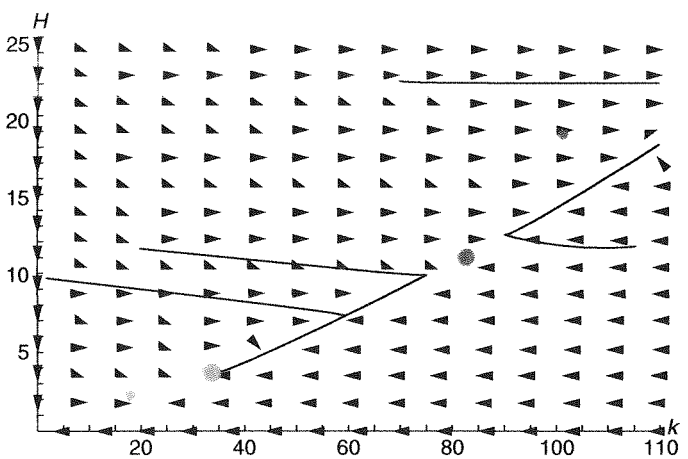


Figure 3.6.4 An increase in the propensity to save.

the points with larger sizes is the new steady states and the other two points with smaller sizes are the old steady states. The two steady states shift as follows:

$$(k_1, H_1) : (17.858, 2.226) \Rightarrow (14.758, 1.718),$$

$$(k_2, H_2) : (101.401, 19.509) \Rightarrow (230.783, 53.417).$$

We see that the new stable steady state has higher levels of human capital and per capita wealth; but the new unstable steady state has lower levels of k and H. Figure 3.6.4 depicts the impact of change in λ on the dynamics of the system.

3.7 The MIPF approach

If money directly affects the utility functions, in equilibrium agents would choose to hold positive amount of money so that money would be positively valued. Since Sidrauski (1967a) introduced money in the utility function, this approach has been used widely to examine a variety of issues in monetary economics. This approach has been criticized for solving the problem of creating positive value for money by simply assuming the problem away.[33] As money is intrinsically useless (as with a paper currency), one might wonder how utility from holding money can strongly affect macroeconomic economic performance. Under certain circumstances money might yield utility, even if never actually spent, if it is held for insurance purchases.[34] Nevertheless, the typical attitude is illustrated by Walsh (2003: 46): "putting money in the utility function may be a useful shortcut for ensuring that there is a demand for money, but it is just a shortcut."

This section has so far examined monetary growth economies with MIUF approach. Another alternative is models with money in production function (MIPF). Like in the MIUF approach, the MIPF approach deals with the role of transactions services provided by money. The entrance of money into production function is based on the assumption that money frees resources for the production of commodities.[35] There are also some empirical studies which examine the role of real money balances in the aggregate production function.[36] As mentioned in Benhabib *et al.* (1991), in industrialized countries firms hold a substantial fraction of the money supply. Although it is straightforward to introduce the MIPF approach into the OSG growth model, partly because the introduction is straightforward and partly because few new insights into monetary effects will be obtained, we use the Ramsey approach to illustrate the basic idea in the approach.

It is assumed that money is held by firms to facilitate production and can therefore be considered as a complementary factor to capital

$$y(t) = f(k(t), m(t)), \quad f_k, f_m > 0.$$

The household is to maximize the following utility functional[37]

$$U = \int_0^\infty u(c(t)) e^{-\rho t} dt,$$

where $c(t)$ and $m(t)$ are per capita consumption and real money balances and ρ is the rate of time preference. Real per capita accumulation is given by

$$\dot{a}(t) = f(k(t), m(t)) + x(t) - c(t) - na(t) - \pi(t)m(t),$$

where $a(t) \equiv k(k) + m(t)$ and $x(t)$ is the government transfers. The present value Hamiltonian is

$$H = u(c) + \lambda[f(k, m) + x - c - na - \pi m] + \bar{\lambda}(a - k - m),$$

where $\lambda(t)$ and $\bar{\lambda}(t)$ are respectively the costate variable associated with the flow constraint and the Lagrangian multiplier associated with the stock constraint. Necessary conditions for an internal solution are

$$\dot{\lambda} = (\rho + n)\lambda - \bar{\lambda},$$

$$\lambda f_k(k, m) - \bar{\lambda} = 0,$$

$$\lambda f_m(k, m) - \lambda\pi - \bar{\lambda} = 0,$$

$$u_c - \lambda = 0.$$

We see that at steady state we have

$$f_k(k, m) = \rho + n,$$

$$f_m(k, m) - \pi - \rho - n = 0.$$

Take derivatives of the above equations with respect to μ

$$f_{kk}\frac{dk}{d\mu} + f_{km}\frac{dm}{d\mu} = 0,$$

$$f_{km}\frac{dk}{d\mu} + f_{mm}\frac{dm}{d\mu} - 1 = 0,$$

We solve the impact of an increase in inflation on capital accumulation as follows

$$\frac{dk}{d\mu} = -\frac{f_{km}}{f_{mm}f_{kk} - (f_{km})^2}.$$

If the production function is concave, and $f_{km} > 0$, then a rise in inflation reduces capital holdings. In general the sign of $f_{mm}f_{kk} - (f_{km})^2$ is ambiguous.

3.8 Further issues

The approach examined in this section can be extended and generalized in different ways. For instance, time distribution between leisure and work is constantly

changing as living conditions and the work environment change. The implications of economic rationality in the allocation of time have been made explicit in a formal and rigorous theory of the subject since Becker published his seminal work in 1965.[38] There have been studies on the interdependence between the productivity of a person's labor, his income, the price of leisure time and time allocation. There is an immense body of empirical and theoretical literature on economic growth with time distribution between home and non-home economic and leisure activities.[39] In Appendix A3.4, we introduce a monetary growth model with endogenous time.[40]

APPENDIX

A.3.1 Proving lemma 3.6.1

First, we find conditions such that $\Phi(H) = 0$ has positive solutions. We exclude the case of $x_e = x_i = 0$. It is easy to check that if $x_e > 0$ and $x_i > 0$ (which guarantees $\Phi(0) < 0, y\Phi(\infty) > 0$ and $\Phi'(H) > 0$ for $H > 0$) or $x_e < 0$ and $x_i < 0$ (which guarantees $\Phi(0) > 0$, $\Phi(\infty) < 0$ and $\Phi'(H) < 0$ for $H > 0$), the system has a unique positive equilibrium point. We now show that in the cases of $x_e > 0$ and $x_i < 0$ (or $x_e < 0$ and $x_i > 0$), the system has two equilibrium points under some additional conditions.

We just prove one case, $x_e > 0$ and $x_i < 0$. The other case can be similarly checked. Since $x_e > 0$ and $x_i < 0$, $\Phi(0) > 0$ and $\Phi(\infty) > 0$. This implies that $\Phi(H) = 0$ has no solution, one solution, or multiple solutions. Since

$$H\Phi' = x_e\Phi_e + x_i\Phi_i,$$

where Φ_e and Φ_i are positive, we see that $\Phi'(H)$ may be either positive or negative, depending upon the values of x_e and x_i. If $\Phi(H) = 0$ has more than two positive solutions, there are at least two positive values of H such that $\Phi'(H) = 0$. Since $dH\Phi'/dH > 0$ strictly holds for $H > 0$, it is impossible for $\Phi'(H) = 0$ to have more than one solution. Accordingly, $\Phi(H) = 0$ has no one, one or two solutions. A necessary and sufficient condition for the existence of two equilibrium points is that there exists a value of H^* such that $\Phi(H^*) < 0$. It should be noted that it is straightforward to show that as the parameters vary, the system may experience bifurcations from a single equilibrium point to two equilibrium points. Although the bifurcation analysis for this model is not difficult and it is not difficult to simulate the conditions,[41] the analysis is tedious and provides few new insights. We omit this issue.

A.3.2 Fluctuations in a MIUF model in the Ramsey approach

This section is to identify endogenous fluctuations in a monetary economy.[42] The economy is inhabited by an infinitely lived representative agent with perfect

foresight, who maximizes $\sum_{t=0}^{\infty} \beta^t U(c(t), m(t))$, subject to the flow budget constraint

$$M(t) = P(t)(y - c(t)) + H(t) + M(t-1), M(-1) \text{ given.}$$

where $U : R^{+2} \to R$ is the one period utility function, which is strictly concave, increasing in both arguments, $\beta \in (0, 1)$ is the discount factor; y is the constant endowment of perishable consumption good; $c(t)$ is consumption level, $m(t) = M(t)/P(t)$ is real balance, $M(t)$ is nominal money holdings, and $P(t)$ the price level. The agent considers $\{P(t)\}_t^{\infty}$ to be independent of his own money holdings. At the beginning of period t, the agent receives $H(t)$ units of paper money from the government through a "helicopter drop," also considered independent of his money holdings. The money supply grows at the rate $\mu > \beta$, which implies $H(t) = (\mu - 1)M(t-1)$. Thus the markets clear when $M(t) = \mu^t M_0$ and $c(t) = y$ for all t. An equilibrium point of the economy is given by a nonnegative sequence of real balances satisfying

$$\beta U_c(y, m(t+1))m(t+1) = \mu m(t)\left[U_c(y, m(t)) - U_m(y, m(t))\right], \quad \text{(A.3.2.1)}$$

as well as the transversality condition: $\lim_{t\to\infty} \beta^t U_c(y, m(t))m(t) = 0$. The steady state equilibrium is $m(t) = m^*$ or $P(t) = \mu^t (M_0/m^*)$ for all t where $m^* > 0$ satisfies

$$(\mu - \beta) U_c(y, m^*) = \mu U_m(y, m^*),$$

exists uniquely.[43]

For simplicity, assume that the utility function is taken on the following form

$$U(c, m) = \begin{cases} -\dfrac{[g(c)m]^{-(1+\eta)}}{1+\eta}, & \text{if } \eta \neq -1, \\ \log g\ (c) + \log m, \end{cases} \quad \text{(A.3.2.2)}$$

where $g > 0, g' > 0$, and $\sup\left[gg''/g'^2\right] < 1$, and η is a parameter satisfying

$$(\eta + 2)\left[2 - \sup\left(\frac{gg''}{g'^2}\right)\right] > 1.$$

The elasticity of intertemporal substitution of real balances is equal to $\sigma = (\eta + 2)^{-1}$. Substituting (A.3.2.2) into (A.3.2.1) yields

$$p(t+1) = (1+\delta)^{1/\eta} p(t)(1 - p(t))^{1/\eta} \equiv F(p(t)), \quad p(t) \in (0, 1), \text{ for all } t, \quad \text{(A.3.2.3)}$$

where $\delta \equiv \mu/\beta - 1 > 0$ and $p(t) \equiv g(y)/\left[g'(y)m(t)\right]$. For the new variable p, the transversality condition becomes $\lim_{t\to\infty} \beta^t p^{\eta}(t) = 0$. The unique steady state

equilibrium is $p^* = \delta/(1+\delta)$. We also rule out the case of $\eta = 0$. In the remaining part of this section, we require: $\eta > 0$.[44] If $\eta = 1$, then

$$F = (1+\delta)p(t)(1-p(t)), \quad p(t) \in (0,1).$$

This is the well-studied logistic map. There are periodic, aperiodic and chaotic solutions to this system.

It is straightforward to check the following properties of F: (1) $F(0) = F(1) = 0$; (2) F has a single peak at $\tilde{p} = \eta/(1+\eta)$; F is strictly increasing on $[0, \tilde{p})$, and strictly decreasing on $(\tilde{p}, 1]$; (3) $F'(0) = (1+\delta)^{1/\eta} > 1$; (4) $F'(p^*) = 1 - \delta/\eta$; and (5) Let $\Delta(\eta) \equiv \eta^{-\eta}(1+\eta)^{1+\eta} - 1$. If $\delta \leq (>)\Delta(\eta), F(\tilde{p}) \leq (>)1$; F maps $[0, 1]$ into itself if $\delta \leq \Delta(\eta)$; F maps $(0, 1)$ into itself if $\delta < \Delta(\eta)$. The function $\Delta(\eta)$ defined on $(0, \infty)$ is strictly increasing, $\lim_{\eta \to \infty} \Delta(\eta) = 0$ and $\Delta(\eta) > 0$ for all η.

Proposition A.3.2.1

i If $0 < \delta \leq 2\eta$, then for all $p_0 \in (0,1)$, $\lim_{t \to \infty} F^t(p_0) = p^*$;

ii If $0 < 2\eta \leq \delta$, then a period-2 cycle exists, and the set $p_0 \in (0,1)$ such that $F^t(p_0)$ converges, N, is at most countable. Furthermore, if $2\eta \leq \delta < \Delta(\eta)$, the set of initial prices that lead to equilibria along which the price level fluctuates forever, N^*, is of full Lebesgue measure.

Proof

i Introduce $G(p) \equiv \{F^2(p)/p\}^{\eta}$. Clearly, $G(p^*) = 1$. If $G(\hat{p}) = 1$ for some $\hat{p} \in (0,1) \setminus \{p^*\}$, then there exists a period-2 cycle of F. Through the definitions of F and G, we have

$$G(p) = (1+\delta)^2(1-p)(1-F(p)),$$

$$G'(p) = (1+\delta)^2\left[\{(1+\delta)(1-p)\}^{1/\eta}\left\{\left(2+\frac{1}{\eta}\right)p - 1\right\} - 1\right],$$

$$G''(p) = (1+\delta)^{2+1/\eta}\left(1+\frac{1}{\eta}\right)(1-p)^{1/\eta-1}\left\{2-\left(2+\frac{1}{\eta}\right)p\right\}.$$

Let $\overline{p} = 2\eta/(1+2\eta)$. We see that G' is strictly increasing in $(0, \overline{p})$, and strictly decreasing $(\overline{p}, 1]$, and $G'(\overline{p}) \leq 0$. Thus $G'(p) < 0$ in $(0,1) \setminus \{\overline{p}\}$, or G is strictly decreasing in $(0, 1)$. Therefore, p^* is the only solution of $G(p) = 1$. This implies that F has no period-2 cycle. From the theorem of Coppel,[45] which states that if f is a continuous map of a compact interval to itself that has no period-2, the sequence $\{f^t(x)\}$ converges to a fixed point of f for every x in the interval. As no $p \in (0,1)$ can approach 0 asymptotically, we have $\lim_{t \to \infty} F^t(p) = p^*$ for every p in $(0, 1)$.

ii For the existence of period-2 cycles, it is sufficient to show that $G(p) = 1$ has a solution in $(0,1) \setminus \{p^*\}$. As $G(0^+) > 0$ and $G(1) = 0$, the Intermediate Value Theorem implies that there exists $p_c \in (0, p^*)$ such that $G(p_c) = 1$ if

$$G'(p^*) = (1+\delta)(\delta/\eta - 2) > 0.$$

To prove countability of N, let p be a point in N and p^{∞} be the limit point of the sequence starting at p. From the continuity of F,

$$p^{\infty} = \lim_{t\to\infty} F^{t+1}(p) = F\left(\lim_{t\to\infty} F^{t}(p)\right) = F(p^{\infty}),$$

we see $p^{\infty} = 0$ or $p^{\infty} = p^{*}$. Since $0 < 2\eta < \delta$ implies that both 0 and p^{*} are locally unstable, $\lim_{t\to\infty} F^{t}(p)$ cannot approach them asymptotically. Therefore, $N = \left\{q \in (0,1) \middle| F^{T}(q) = 0 \text{ or} p^{*} \text{for a finite} T\right\}$ which is at most countable, since Property (2) of F implies that, for any z, there are at most two x's that solve $z = F(x)$. The second half follows from the fact that if $\delta < \Delta(\eta), N^{*} = (0,1)\backslash N$.

Proposition A.3.2.2
For any $\eta > 0$, there exists a value $\Delta^{*}(\eta)$ satisfying $2\eta < \Delta^{*}(\eta) < \Delta(\eta)$ such that a period 3 cycle of F exists if $\delta > \Delta^{*}(\eta)$.

Proof
Introduce

$$G(p) \equiv \left\{F^{3}(p)/p\right\}^{\eta}.$$

Then $G(0^{+}) > 1$ and $G(p^{*}) = 1$. From the Intermediate Value Theorem, it is sufficient to show that there is a $p_c \in (0, p^{*})$ such that $G(p_c) < 1$. If $\delta > \Delta(\eta)$, then there exists a $p_c \in (0, p^{*})$ such that $F(p_c) = 1$ so that $F^{2}(p_c) = F^{3}(p_c) = 0$ or $G(p_c) < 1$. From the continuity of G on δ, there exists a $\Delta^{*}(\eta) < \Delta(\eta)$ such that $G(p_c) < 1$. That $2\eta < \Delta^{*}(\eta)$ follows from Proposition A.3.2.1.

The proposition guarantees the existence of chaos.

A.3.3 Growth with heterogeneous households[46]

It is straightforward to extend the two-group model of the previous section to multiple groups of consumers. We may generally assume that the population is N_0 and the population can be classified into M groups, indexed by j, according to their preferences, wealth, human capital, and social status. We have $N_0 \geq M$.[47] Two extreme cases are $M = N_0$ and $M = 1$. Let the number of group j be N_j. The aggregated labor force, N, is given by

$$N_0 = \sum_{j=1}^{M} z_j N_j,$$

where z_j is the level of human capital or the work time of group j. The neoclassical production function is $F(t) = F(K(t), N)$. Let K, r, w_j, and $\delta_k (0 \leq \delta_k < 1)$ be defined as before. The marginal conditions are given by

$$r(t) + \delta_k = f'(k(t)),$$
$$w_j(t) = z_j w(t), w(t) = f(k(t)) - k(t) f'(k(t)). \qquad \text{(A.3.3.1)}$$

The government levies no taxes. Money is introduced by assuming that a central bank distributes at no cost to the population a per capita amount of fiat money $M(t) > 0$. The scheme according to which the money stock evolves over time is deterministic and known to all agents. With μ being the constant net growth rate of the money stock, $M(t)$ evolves over time according to

$$\dot{M}(t) = \mu M(t), \quad \mu > 0.$$

The government expenditure in real terms per capita, $g(t)$, is given by

$$g(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t), \tag{A.3.3.2}$$

where $P(t)$ is the price of the output good and $m(t)$ stands for the real value of money per capita measured in units of the output good. The inflation rate, $\pi(t)$, is given by

$$\pi(t) \equiv \frac{\dot{P}(t)}{P(t)} = \mu - \frac{\dot{m}(t)}{m(t)}. \tag{A.3.3.3}$$

Household j's disposable income in real terms, $\hat{y}_j(t)$, is

$$\hat{y}_j(t) = r(t)k_j(t) + w_j(t) + \mu m(t) - \pi(t)m_j(t) + a_j(t), \tag{A.3.3.4}$$

where $a_j(t) \equiv k_j(t) + m_j(t)$.

We assume that at each point of time consumers' preferences over money, consumption and saving can be represented by the following utility function

$$U_j(t) = m^{\varepsilon_0}(t)c^{\xi_0}(t)s^{\lambda_0}(t), \quad \varepsilon_0, \xi_0, \lambda_0 > 0,$$

where ε_{0j} is called household j's propensity to hold money, ξ_{0j} propensity to consume, and λ_{0j} propensity to own wealth. The budget constraint is given by

$$(1 + r(t))m_j(t) + c_j(t) + s_j(t) = \hat{y}_j(t).$$

Insert equation (A.3.3.4) in the budget constraints

$$(r + \pi)m_j + c_j + s_j = \bar{y}_j \equiv (1 + r)k_j + w_j + \mu m. \tag{A.3.3.5}$$

Consumers' problem is to choose current money, consumption, and savings. Maximizing $U_j(t)$ subject to the budget constraint (A.3.3.5) yields

$$\pi m_j = \varepsilon_j \bar{y}_j, \quad c_j = \xi_j \bar{y}_j, \quad s_j = \lambda_j \bar{y}_j, \tag{A.3.3.6}$$

where

$$\varepsilon_j \equiv \rho_j \varepsilon_{0j}, \quad \xi_j \equiv \rho_j \xi_{0j}, \quad \lambda_j \equiv \rho_j \lambda_{0j}, \quad \rho_j \equiv \frac{1}{\varepsilon_{0j} + \xi_{0j} + \lambda_{0j}}.$$

The real wealth changes as follows

$$\dot{a}_j(t) = s_j(t) - a_j(t). \tag{A.3.3.7}$$

The total demand for money is equal to the total supply

$$N_0 m(t) = \sum_{j=1}^{J} m_j(t) N_j. \tag{A.3.3.8}$$

The assumption of full employment of capital is given by

$$K(t) = \sum_{j=1}^{J} k_j(t) N_j,$$

that is

$$k(t) = \sum_{j=1}^{J} n_j k_j(t), \tag{A.3.3.9}$$

where $n_j \equiv N_j/N$. This equation expresses $k(t)$ as a unique function of $\{k_j(t)\}$, where $\{k_j(t)\}$ denotes the per-capita distribution of capital wealth among the households. We have thus built the model.

The dynamics and equilibrium

First, we need to find equations which describe motion of the dynamic system. For simplicity, we omit time. First, multiplying equation $(r+\pi)m_j = \varepsilon_j \bar{y}_j$ in (A.3.3.6) by N_j and then adding up the resulted J equations, we obtain

$$(r+\pi)\sum_{j=1}^{J} N_j m_j = \sum_{j=1}^{J} \varepsilon_j N_j \bar{y}_j.$$

Substituting equation (A.3.3.8) into the above equation yields

$$(r+\pi) = \frac{1}{m}\sum_{j=1}^{J} \bar{\varepsilon}_j \bar{y}_j, \tag{A.3.3.10}$$

where $\bar{\varepsilon}_j \equiv \varepsilon_j N_j/N_0$. Insert equation (A.3.3.3) in the above equation

$$\dot{m} = \Omega_m\left(\{k_j\}, m\right) \equiv (\mu + r)m - \sum_{j=1}^{J} \bar{\varepsilon}_j \bar{y}_j. \tag{A.3.3.11}$$

As k is a unique function of $\{k_j\}$, from equations (A.3.3.1) we see that r and w_j can also be expressed as unique functions of $\{k_j\}$. From the definition of $\bar{y}_j(t)$, $\bar{y}_j$ are functions of $\{k_j\}$ and m. From $\pi m_j = \varepsilon_j \bar{y}_j$ and equation (A.3.3.10), we have

$$m_j = \Omega_j(\{k_j\}, m) \equiv \varepsilon_j \bar{y}_j m Y, \tag{A.3.3.12}$$

where

$$Y \equiv \frac{1}{\sum_{j=1}^{J} \bar{\varepsilon}_j \bar{y}_j}.$$

Substituting $s_j = \lambda_j \bar{y}_j$ and $a_j = k_j + m_j$ into equations (A.3.3.13), we have

$$\dot{k}_j = (1+r)\lambda_j k_j + \lambda_j w_j + \lambda_j \mu m - k_j - \Omega_j - \dot{m}_j. \tag{A.3.3.13}$$

Taking derivatives of equations (A.3.3.12) with respect to t yields

$$\frac{\dot{m}_j}{m_j} = \frac{\Omega_m}{m} + \frac{\dot{\bar{y}}_j}{\bar{y}_j} - Y \sum_{q=1}^{J} \bar{\varepsilon}_q \dot{\bar{y}}_q, \tag{A.3.3.14}$$

where

$$\dot{\bar{y}}_j = (k_j - z_j k) f'' \sum_{j=1}^{J} n_j \dot{k}_j + (1+r)\dot{k}_j + \mu \Omega_m,$$

in which we use equations (A.3.3.1) and (A.3.3.9). It can be seen that from equations (A.3.3.13) and (A.3.3.14), we can determine $\dot{k}_j$ by the following equation

$$A_{J\times J}(\{k_j\}, m)[\dot{k}_j]_{1\times J} = B_{1\times J}(\{k_j\}, m),$$

where $A_{J\times J}$ and $B_{1\times J}$ are respectively $J\times J$ and $1\times J$ vectors and they are functions of $\{k_j\}$ and m.[48] If the inverse of $A_{J\times J}$ exists, then we have

$$[\dot{k}_j]_{1\times J} = A^{-1}_{J\times J} B_{1\times J}. \tag{A.3.3.15}$$

Lemma A.3.3.1
The motion of the economic system is described by $J+1$ differential equations with $\{k_j(t)\}$ and $m(t)$ as variables

$$[\dot{k}_j]_{1\times J} = A^{-1}_{J\times J}(\{k_j\}, m) B_{1\times J}(\{k_j\}, m),$$
$$\dot{m} = \Omega_m(\{k_j\}, m). \tag{A.3.3.16}$$

All the other variables are determined as functions of $\{k_j(t)\}$ and $m(t)$ by the following procedure: $k(t)$ by (A.3.3.9) $\rightarrow m_j(t), j = 1, \ldots, J$, by (A.3.3.12) $\rightarrow$

$a_j(t) = k_j(t) + m_j(t) \rightarrow w_j(t)$ and $r(t)$ by (A.3.3.1) $\rightarrow K(t) = k(t)N \rightarrow \pi(t)$ by (A.3.3.10) $\rightarrow \bar{y}_j(t)$ by (A.3.3.5) $\rightarrow c_j(t)$ and $s_j(t)$ by (A.3.3.6).

As the lemma provides a computational process to follow the motion of all the variables in the system, it is straightforward to simulate the model.

A.3.4 Endogenous leisure and money

We now introduce endogenous time into the OSG model proposed in Section 3.3.[49] Almost all the aspects of the following model are the same as in Section 3.3. Let $N(t)$ stand for the flow of labor services used at time t for production. The total labor force, $N(t)$, is now given by $N(t) = T(t)N_0$, where $T(t)$ is the work time of a representative household and N_0 (= 1) is the total labor force. Introduce $f(k(t)) \equiv F(k(t), 1)$, where $k(t) \equiv K(t)/N(t)$. The production sector chooses the two variables $K(t)$ and $N(t)$ to maximize its profit. The marginal conditions are given by

$$r(t) + \delta_k = f'(k(t)), \quad w(t) = f(k(t)) - k(t)f'(k(t)). \tag{A.3.4.1}$$

Let $\bar{k}(t)(\equiv K(t)/N_0)$ stand for per capita physical wealth. According to the definition of $k(t)$ and $\bar{k}(t)$, we have $\bar{k}(t) = k(t)T(t)$. Money is introduced by assuming that a central bank distributes at no cost to the population a per capita amount of fiat money $M(t) > 0$. The scheme according to which the money stock evolves over time is deterministic and known to all agents. With μ being the constant net growth rate of the money stock, $M(t)$ evolves over time according to

$$\dot{M}(t) = \mu M(t), \quad \mu > 0.$$

The government expenditure in real terms per capita, $\tau(t)$, is given by

$$\tau(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t). \tag{A.3.4.2}$$

The representative household receives $\mu m(t)$ units of paper money from the government through a "helicopter drop," also considered independent of his money holdings.

The disposable income in real terms, $\hat{y}(t)$, is

$$\hat{y}(t) = r(t)\bar{k}(t) + w(t)T(t) - \pi(t)m(t) + \tau(t) + a(t),$$

where $a(t) \equiv \bar{k}(t) + m(t)$. Denote $T_h(t)$ the leisure time at time t and the (fixed) available time for work and leisure by T_0. The time constraint is expressed by

$$T(t) + T_h(t) = T_0.$$

Substituting this time constraint into the disposable income yields

$$\hat{y}(t) = r(t)\bar{k}(t) + w(t)T_0 - w(t)T_h(t) - \pi(t)m(t) + \tau(t) + a(t). \tag{A.3.4.3}$$

We assume that at each point of time the consumer's utility function for holding money, consuming leisure and goods, and making savings is to be represented by the following utility function

$$U(t) = m^{\varepsilon_0}(t)T_h^{\sigma_0}(t)c^{\xi_0}(t)s^{\lambda_0}(t), \quad \varepsilon_0, \sigma_0\xi_0, \lambda_0 > 0.$$

where ε_0 is called propensity to hold money, σ_0 and ξ_0 propensities to consume leisure and goods, and λ_0 propensity to own wealth. The budget constraint is given by

$$(1+r(t))m(t)+c(t)+s(t)=\hat{y}(t).$$

Insert equation (A.3.4.3) in the budget constraint

$$(r+\pi)m+wT_h+c+s=\bar{y}\equiv(1+r)\bar{k}+wT_0+\tau. \tag{A.3.4.4}$$

Here, we may interpret the variable $\bar{y}(t)$ as the potential disposable income as it is the disposable income $\hat{y}(t)$ when all the available time is spent on work (i.e. $T = T_0$). Consumers' problem is to choose current money, consumption, and savings in such a way that utility levels are maximized. Maximizing $U(t)$ subject to the budget constraint (A.3.4.4) yields

$$(r+\pi)m=\varepsilon\bar{y}, \quad wT_h=\sigma\bar{y}, \quad c=\xi\bar{y}, \quad s=\lambda\bar{y}, \tag{A.3.4.5}$$

where

$$\varepsilon\equiv\rho\varepsilon_0, \quad \sigma\equiv\rho\sigma_0, \quad \xi\equiv\rho\xi_0, \quad \lambda\equiv\rho\lambda_0, \quad \rho\equiv\frac{1}{\varepsilon_0+\sigma_0+\xi_0+\lambda_0}.$$

The real wealth changes as follows

$$\dot{a}(t)=s(t)-a(t). \tag{A.3.4.6}$$

We have thus built the model. We now find dynamics of capital accumulation. First, we need to find equations which describe motion of the dynamic system. From the definition of $\bar{y}$, $\pi=\mu-\dot{m}/m$ and $(r+\pi)m=\varepsilon\bar{y}$, we obtain

$$\dot{m}=(r+\mu-\varepsilon\mu)m-(1+r)\varepsilon kT-\varepsilon wT_0, \tag{A.3.4.7}$$

where we use $\tau=\mu m$. As r and w are functions of k, the right hand side of equation (A.3.4.7) is a function of T, k, and m.

According to the definition of $s(t)$, the change in the household's wealth is given by

$$\dot{\bar{k}}=\lambda\bar{y}-\bar{k}-m-\dot{m}. \tag{A.3.4.8}$$

From the definition of $\bar{y}(t)$ and equations (A.3.4.1), we obtain

$$\bar{y} = \left(f'(k) + \delta\right) kT + \left(f(k) - kf'(k)\right) T_0 + \varepsilon m, \tag{A.3.4.9}$$

where $\delta \equiv 1 - \delta_k$. Substituting equation (A.3.4.7) into $w(t) T_h(t) = \sigma \bar{y}(t)$ yields

$$T = \left[\left(f - kf'\right)(1-\sigma) T_0 - \sigma \varepsilon m\right] \tilde{f}, \tag{A.3.4.10}$$

where we use equation (A.3.4.1) and $T + T_h = T_0$ and

$$\tilde{f}(k) \equiv \frac{1}{f - (1-\sigma) f' k + \delta \sigma k}.$$

We see that T is a function of k and m. From $f > kf'$, we see that $\tilde{f}(k)$ is always positive. We should require that for any meaningful positive k, $0 < T(t) < T_0$. This is guaranteed if

$$f - kf' > \frac{\sigma \varepsilon m}{(1-\sigma) T_0}. \tag{A.3.4.11}$$

From equation (A.3.4.9), we have

$$\lambda \bar{y} - \dot{\bar{k}} - m - \dot{m} = \left(\lambda f' + \lambda \delta - 1\right) kT + \left(f - kf'\right) \lambda T_0 - (1 - \lambda \varepsilon) m - \dot{m}, \tag{A.3.4.12}$$

As $\dot{m}$ is a function of T, k, and m, and T is a function of k and m, the left-hand side of equation (A.3.4.12) is a function of k and m.

Take derivatives of $\bar{k} = Tk$ with respect to t

$$\dot{\bar{k}} = \dot{T} k + T \dot{k}. \tag{A.3.4.13}$$

This equation expresses how the change in per-capita wealth is related to the changes in the labor intensity and the labor time. From equation (A.3.4.10), we have

$$\dot{T} = \left\{-(1-\sigma) T_0 k f'' \dot{k} - \sigma \varepsilon \dot{m} - \left[f' - (1-\sigma)\left(f' + f'' k\right) + \delta \sigma\right] T \dot{k}\right\} \tilde{f}.$$

Insert the above equation in equation (A.3.4.13)

$$\dot{\bar{k}} = -\sigma \varepsilon k \tilde{f} \dot{m} + \Lambda_0(k, m) \dot{k}, \tag{A.3.4.14}$$

where

$$\Lambda_0(k, m) \equiv \frac{\left[f - f' k - (1-\sigma) f' k + (1-\sigma)\left(f' + f'' k\right) k\right] T - (1-\sigma) T_0 k^2 f''}{f - (1-\sigma) f' k + \delta \sigma k} \tilde{f}.$$

Combining equations (A.3.4.8), (A.3.4.11), and (A.3.4.14), we get a nonlinear differential equation in k and m as follows

$$\dot{k} = \Lambda_1(k,m) + \frac{\left(\sigma\varepsilon\tilde{f}k - 1\right)\dot{m}}{\Lambda_0(k,m)}, \tag{A.3.4.15}$$

where

$$\Lambda_1(k,m) \equiv \frac{\left(\lambda f' + \lambda\delta - 1\right)kT + \left(f - kf'\right)\lambda T_0 - (1-\lambda\varepsilon)m}{\Lambda_0(k,m)}.$$

The differential equation (A.3.4.15) contains two variables, $m(t)$ and $k(t)$. The following lemma says that for any meaning solution of equations (A.3.4.7) and (A.3.4.15), all the other variables in the dynamic system are uniquely determined.

Lemma A.3.4.1
The motion of $m(t)$ and $k(t)$ is determined by the two differential equations

$$\begin{aligned}
\dot{m} &= \left(f' + \delta_0\right)m - \left(f' + \delta\right)\varepsilon kT - \left(f - kf'\right)\varepsilon T_0, \\
\dot{k} &= \Lambda_1(k,m) + \frac{\left(\sigma\varepsilon\tilde{f}k - 1\right)\dot{m}}{\Lambda_0(k,m)},
\end{aligned} \tag{A.3.4.16}$$

where $\delta_0 \equiv \mu - \varepsilon\mu - \delta_k$. For any positive $m(t)$ and $k(t)$, all the other variables are uniquely determined at any point of time by the following procedure: $T(t)$ by equation (A.3.4.10) $\rightarrow T_h(t) = T_0 - T(t) \rightarrow \overline{k}(t) = k(t)T(t) \rightarrow f(t) = f(k(t)) \rightarrow r(t)$ and $w(t)$ by equations (A.3.4.1) $\rightarrow \overline{y}(t)$ by equation (A.3.4.9) $\rightarrow \pi(t), c(t)$ and $s(t)$ by the equations (A.3.4.5) $\rightarrow \tau(t) = \mu m(t) \rightarrow N(t) = N_0T(t) \rightarrow a(t) = \overline{k}(t) + m(t) \rightarrow F(t) = f(t)N(t)$.

The lemma guarantees that the dynamics of the system is determined by the differential equation (A.3.4.16). A steady state of the differential equations is determined by

$$\begin{aligned}
&\left(f' + \delta_0\right)m - \left(f' + \delta\right)\varepsilon\tilde{f}k\left[\left(f - kf'\right)(1-\sigma)T_0 - \sigma\varepsilon m\right] - \left(f - kf'\right)\varepsilon T_0 = 0, \\
&\left(\lambda f' + \lambda\delta - 1\right)\left[\left(f - kf'\right)(1-\sigma)T_0 - \sigma\varepsilon m\right]\tilde{f}k + \left(f - kf'\right)\lambda T_0 \\
&\quad - (1 - \lambda\varepsilon)m = 0,
\end{aligned} \tag{A.3.4.17}$$

where we also use equation (A.3.4.10). From the first equation of (A.3.4.17), we solve

$$m = \left[\frac{1 + (1-\sigma)\left(f' + \delta\right)\tilde{f}k}{\left(f' + \delta_0\right) + \left(f' + \delta\right)\sigma\varepsilon^2 k\tilde{f}}\right]\left(f - kf'\right)\varepsilon T_0 > 0, \quad \forall k > 0. \tag{A.3.4.18}$$

It should be noted that under (A.3.4.18), the constraint (A.3.4.11) becomes

$$f' + \mu - \delta_k > \frac{\sigma\varepsilon^2}{1-\sigma} + \varepsilon\mu > 0. \tag{A.3.4.19}$$

It is necessary to have

$$f' + \mu - \delta_k > 0.$$

Substituting equation (A.3.4.18) into the second equation in (A.3.4.17) yields

$$\Omega(k) \equiv \frac{\lambda + (\lambda f' + \lambda\delta - 1)(1-\sigma)k\tilde{f}}{(1-\lambda\varepsilon) + (\lambda f' + \lambda\delta - 1)\sigma\varepsilon k\tilde{f}} - \frac{\varepsilon + (1-\sigma)(f' + \delta)\varepsilon k\tilde{f}}{(f' + \delta_0) + (f' + \delta)\sigma\varepsilon^2 k\tilde{f}} = 0. \tag{A.3.4.20}$$

The equation, $\Omega(k) = 0$, has a single variable.

A.3.5 A monetary growth model with endogenous population

Another important issue related to growth and money is about endogenous population. As money is an important factor for economic growth and the interdependence between population and economic growth is a classical topic, it is evidently meaningful to examine how money may affect population. This section deals with interactions among growth, population, and money. Although this study does not explore complicated issues related to endogenous population, we introduce a monetary growth model in the MIUF approach for another way of extending our analytical framework.

Petrucci (2003) synthesizes the growth theory with fertility[50] and Sidrauski's model in a compact framework. The basic assumption in the literature of endogenous fertility is that the number of children enters directly into the utility function of consumers. The isolated economy is populated by identical agents who are infinitely-lived, accumulate wealth in the form of money balances and physical capital, behave competitively and have perfect foresight. Money enters utility function. The fertility rate also enters utility function. As mortality rate is zero, the population growth rate is equal to the fertility rate. As most of the variables are defined, as in Sidrauski's model in Section 3.1, we only refine variables which are introduced in this section. The household is to maximize the following utility functional

$$U = \int_0^\infty u(c(t), n(t), m(t))e^{-\rho t}dt,$$

where $n(t)$ is the fertility rate. For simplicity of analysis, we assume that the instantaneous utility function is additively separable in consumption and fertility,

on the one hand, and money holdings, on the other, as follows[51]

$$u(c, n, m) = u_1(c, n) + u_2(m).$$

The sub-utility function, u_1, is twice-continuously differentiable, increasing and strictly concave in its arguments. c and n are normal goods. The sub-utility function, u_2, is twice-continuously differentiable, increasing and strictly concave in its arguments, and the marginal utility of money satisfies

$$sgn\left[u_2'(m(t))\right] = sgn\left[m^* - m(t)\right],$$

where m^* is the satiation level of real money balances. The time and budget constraints are respectively given by

$$T(t) + T_c(n(t)) = T_0,$$

$$c(t) + \dot{m}(t) + \dot{k}(t) = f(k(t), T(t)) + \tau(t) - (n(t) + \pi(t))m(t) - n(t)k(t),$$

where $T_c(n)$ is time spent for child-rearing and $\tau = \mu m$. We require[52]

$$T_c(0) = 0, T_c(n) > 0, \quad T_c'(n) > 0, \quad T_c'(n) > (\leq)0, \quad \forall n > 0.$$

The evolution of real money balances per capita is

$$\dot{m}(t) = (\mu - \pi(t) - n(t))m(t).$$

Different from Sidrauski's model, in Petrucci's model, money superneutrality is no more held. It is shown that a higher monetary growth rate increases fertility, as by reducing real balance holdings it diminishes the opportunity cost of one unit of fertility, which in turn stimulates people to increase fertility and hence results in a reduction of capital intensity, per capita output and consumption. However, the reverse effect on per capita output and capital is matched by an increase in aggregate output and capital growth rates, as these expansion rates coincide with fertility.

4 Money-in-advance approaches

Chapter 2 considered the role of money as a store of value and Chapter 3 introduced money into the utility and production functions. With regard to the MIUF and MIPF approaches, Cass and Shell (1980) objected

> Imbedding money in preferences or technologies does nothing to explain its role as a store of value. Moreover such reduced forms are at best poor proxies for their structural counterparts. Worst of all, to the extent that this maneuver is successful, it is also likely to be misleading.

Although entering money as an argument of utility and production functions has been widely accepted in the literature of monetary economics and they overcome one of the two key criticisms of the original Tobin model – that demand is determined by maximizing utility subject to constraints – these models do not provide adequate explanation of why money is held, and thus fail to address the second key criticism of the Tobin model. In particular, the transactions role of money is not explicitly considered. This chapter discusses the transactions role of money.

In this chapter, money is held because transactions need to be settled by payment via money. Clower (1967) originally proposed a model to incorporate the role of money as a medium of exchange through the so-called cash-in-advance constraint. The basic idea is to explain the role that money plays in carrying out transactions by introducing transaction technology.[1] The approach holds that goods cannot be exchanged for goods and only money can buy goods. In CIA models the role of money as facilitator of transactions is reflected in the rule that no transactions can take place unless the money needed for the transaction is held for some time in advance. Stockman (1981) developed a growth model through CIA constraints. The model predicts that there is long-run superneutrality if only consumption expenditures are subject to a CIA constraint. If investment is also subject to a CIA constraint then steady state capital will fall when the growth rate of money rises. It is worthwhile mentioning that Feenstra (1986) studied the relationship between the MIUF and CIA approaches. He proved that under certain regularity conditions, the maximization problem with money under a CIA constraint may be equivalent to a maximization problem with the MIUF approach. This implies that although it is

sometimes difficult to justify the validity of the MIUF approach, the approach may be considered as an acceptable approximation. It should be also remarked that the equivalence is held under specified conditions. In a recent two-sector growth model with human capital accumulation proposed by Chang (2002), it is demonstrated that the Feenstra-like functional equivalence between the two approaches does not hold.

The first formal models with money as a medium of exchange that facilitate transactions were due to Baumol (1952) and Tobin (1956). As cash balances are typically non-interest bearing, it is costly for individuals to hold money. Individuals choose to hold only the cash they need and save the rest of their disposable income to earn interest. This means that the transactions demand for money should be negatively related to the rate of interest. We now derive Baumol's square-root rule which relates individual's demand for real cash balances, income and rate of interest.[2]

Let us consider an individual with a nominal income of y per month. He spends all the income during the month. If all his income is held as cash, then the average cash balance held over the month is equal to $y/2$. If he withdraws half of the money from the bank at the start of the month and withdraws half at the start of the third week, then the average cash balance is equal to $(1/2)(y/2)$. If he withdraws n times over the month, then his average cash balance is given by

$$m = \frac{1}{2}\left(\frac{y}{n}\right).$$

Consider that the bank offers interest on the funds remaining in the account with a fixed rate of interest, r per month. His average deposit in the bank is $y/2$ if he withdraws infinite times. When he withdraws n times, his average deposit is $y/2$ less $y/2n$. Hence, the net interest earned is

$$r\left(\frac{y}{2} - \frac{1}{2}\frac{y}{n}\right).$$

There are costs in terms of time and trouble of withdrawing cash for the individual. Assume that the average cost of each withdrawal is c. The total cost of withdrawing n times is equal to cn. The individual's profit is thus given by

$$\Pi = r\left(\frac{y}{2} - \frac{1}{2}\frac{y}{n}\right) - cn.$$

The rational individual will choose n in such a way that Π is maximized. The optimal number is

$$n = \sqrt{\frac{yr}{2c}}.$$

Hence, the optimal money holding is given by

$$m = \sqrt{\frac{cy}{2r}}.$$

This is known as Baumol's square-root rule. An individual's demand for real cash balance is positively related to the income and cost of withdrawing and negatively related to the rate of interest.

The above equation describes the demand of individuals. To find out the aggregate money demand, suppose, for instance, that the consumer buys goods from a representative firm and the firm periodically converts its money-holdings into bonds. The firm's pattern of bond and money holdings would follow the pattern exactly complementary to the consumer's pattern. Hence, the firm's demand for money would also be given by $m = \sqrt{cy/2r}$. The aggregate money demand is thus equal to $2m$. Hence, the aggregate demand for real money transactions balances, M/P, is given by

$$\frac{M}{P} = \sqrt{\frac{2cy}{r}}.$$

In the above equation, y is now national real income. The elasticity of aggregate demand for real money balances for transactions purposes with respect to both income and the rate of interest is 0.5 and -0.5 respectively. Different from Keynes' demand for transactions balances which is proportional to income, Baumol's equation shows that the demand is related both to income and the rate of interest.[3]

4.1 A CIA-OSG growth model

This section introduces cash-in-advance into one-sector growth model with capital. The variables, N, $K(t), F(K(t), N), y(t) = f(k(t)), k(t), M(t), P(t), m(t)$ and δ_k are defined as in Section 2.2. The marginal conditions for the production sector are

$$r(t) + \delta_k = f'(k(t)), \quad w(t) = f(k(t)) - k(t) f'(k(t)). \tag{4.1.1}$$

We fix $N = 1$. We have

$$\dot{M}(t) = \mu M(t), \quad \mu > 0,$$

$$g(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t). \tag{4.1.2}$$

When deciding about the composition of their portfolios, the household knows in advance that a certain fraction of consumption needs to be financed by payment

in cash. Assume that cash has to be held in advance of purchasing goods. The liquidity constraint of the household is formed as[4]

$$M(t) \geq \chi P(t)c(t), \tag{4.1.3}$$

where χ is a positive parameter.[5] We require $0 < \chi < 1$.

From $M = mP$, we have

$$\pi(t) = \frac{\dot{P}(t)}{P(t)} = \mu - \frac{\dot{m}(t)}{m(t)}.$$

The disposable income is given by

$$\hat{y}(t) = r(t)k(t) + w(t) + a(t) - \pi(t)m(t), \tag{4.1.4}$$

where $a(t) \equiv k(t) + m(t)$. The budget constraint is given by $c(t) + s(t) = \hat{y}(t)$. Insert equation (4.1.4) in the budget constraint

$$(1 + \chi\pi(t))c(t) + s(t) = \bar{y}(t) = y(t) + a(t),$$

where we use $m(t) = \chi c(t)$. We specify the utility function in the Cobb-Douglas form

$$U(t) = c^{\xi_0}(t)s^{\lambda_0}(t), \quad \xi_0, \lambda_0 > 0.$$

The optimal solution is

$$c(t) = \frac{\xi\bar{y}(t)}{1 + \chi\pi(t)}, \quad s(t) = \lambda\bar{y}(t), \tag{4.1.5}$$

where

$$\xi \equiv \frac{\xi_0}{\xi_0 + \lambda_0}, \quad \lambda \equiv \frac{\lambda_0}{\xi_0 + \lambda_0}.$$

The wealth accumulation is given by

$$\dot{a}(t) = s(t) - a(t). \tag{4.1.6}$$

The balance of demand and supply in the commodity market is given by

$$S + C + \delta_k K = F + K.$$

It can be shown that this equation is redundant in the sense that it can be derived from the other equations in the system.

The dynamics and steady state

From the definitions of $\bar{y}(t)$ and $a(t)$, we have

$$\bar{y}(t) = f(k(t)) + \delta k(t) + m(t). \tag{4.1.7}$$

where we use $y(t) = f(k(t)) - \delta_k k(t)$ and $\delta \equiv 1 - \delta_k$. Assuming inequality (4.1.3) to be at strict equality, we have

$$m(t) = \chi c(t) = \frac{\chi \xi \bar{y}(t)}{1 + \chi \pi(t)}. \tag{4.1.8}$$

Inserting $\pi(t) = \mu - \dot{m}(t)/m(t)$ in equation (4.1.8), we have

$$\dot{m}(t) = \mu_0 m(t) - \xi f(k(t)) - \delta \xi k(t), \tag{4.1.9}$$

where we use equation (4.1.7) and

$$\mu_0 \equiv \frac{1}{\chi} + \mu - \xi.$$

We require $\mu_0 > 0$, otherwise $m(t)$ will fall monotonically as m, f, and k are positive. As this requirement means

$$1 + \chi\mu > \chi\xi,$$

where ξ is equal to the percentage of the total potential disposable income located to consumption, we see that this inequality simply implies that the demand for money is not larger than the potential disposable income. This requirement is satisfied if χ is not large.

Substituting $s = \lambda\bar{y}$ and $a = k + m$ into (4.1.6) yields

$$\dot{k}(t) = \lambda f(k(t)) - (\xi + \lambda\delta_k) k(t) - \xi m(t) - \dot{m}(t). \tag{4.1.10}$$

We have established the model. The dynamics are given by equations (4.1.9) and (4.1.10). The system contains two variables, $k(t)$ and $m(t)$. For any positive $k(t)$ and $m(t)$, the other variables are determined as follows: $\bar{y}(t)$ by (4.1.7) $\rightarrow$ $P(t)$ by (4.1.2) $\rightarrow$ $\pi(t) = \dot{P}(t)/P(t) \rightarrow a(t) = k(t) + m(t) \rightarrow r(t)$ and $w(t)$ by (4.1.1) $\rightarrow c(t)$ and $s(t)$ by (4.1.5).

By equations (4.1.9) and (4.1.10), at steady state we have

$$\lambda f(k) - (\xi + \lambda\delta_k) k - \xi m = 0,$$

$$\mu_0 m - \xi f(k) - \delta\xi k = 0. \tag{4.1.11}$$

From the second equation in (4.1.11), we solve

$$m = \frac{\xi f(k) + \delta \xi k}{\mu_0}. \tag{4.1.12}$$

This equation determines a unique relation between k and m. Substituting equation (4.1.12) into (4.1.11) yields

$$\chi_1 f(k) - \chi_2 k = 0, \tag{4.1.13}$$

where

$$\chi_1 \equiv \frac{\lambda \mu_0}{\xi^2} - 1 = \frac{\lambda + \lambda \mu \chi - \xi \chi}{\xi^2 \chi}, \quad \chi_2 \equiv (\xi + \lambda \delta_k) \frac{\mu_0}{\xi^2} + \delta > 0.$$

For the system to have a positive equilibrium k, we require $\chi_1 > 0$, that is

$$\lambda + \lambda \mu \chi > \xi \chi. \tag{4.1.14}$$

From the properties of f and $\chi_1 > 0$ and $\chi_2 > 0$, we see that the equation, $\chi_1 f(k) - \chi_2 k = 0$, has a unique solution. From equation (4.1.12), we have m. To examine stability properties of the unique equilibrium point, we calculate the two eigenvalues. From differential equations (4.1.10) and (4.1.9) and their equilibrium equations (4.1.12) and (4.1.13), we calculate the two eigenvalues, ϕ_1 and ϕ_2, as follows

$$\phi_{1,2} = \frac{f_1 \pm \sqrt{f_1^2 - 4 f_0}}{2}, \tag{4.1.15}$$

where $f_1 \equiv \mu_0 + f' - \delta_k$ and

$$f_0 \equiv (f' - \delta_k) \mu_0 - \xi (\xi + \mu_0)(f' + \delta).$$

We now show that f_0 is negative under $\chi_1 > 0$. Using $\xi + \lambda = 1$, we have

$$\begin{aligned} f_0 &= f' (\lambda \mu_0 - \xi^2) - \xi (\xi + \mu_0) - \delta_k (\lambda \mu_0 - \xi^2) \\ &= \frac{\xi^2 \chi_2 k f'}{f} - \xi (\xi + \mu_0) - \delta_k (\lambda \mu_0 - \xi^2), \end{aligned}$$

where we also use

$$\lambda \mu_0 - \xi^2 = \xi^2 \chi_2 k / f,$$

which is obtained from equation (4.1.13) and the definition of χ_1. Using $k f'/f < 1$ and the definition of χ_1, from the above equation we have

$$f_0 < \xi^2 \chi_2 - \xi (\xi + \mu_0) - \delta_k (\lambda \mu_0 - \xi^2) = 0.$$

As $f_0 < 0$, from (4.1.15) we conclude that one eigenvalue is positive and the other one is negative. Accordingly, the unique equilibrium point is a saddle point. In summary, we have the following proposition.

Proposition 4.1.1
If (4.1.15) is satisfied, then the monetary economy has a unique saddle equilibrium point.

Comparative statics analysis

We now examine effects of changes in some parameters on the equilibrium point. It is straightforward to check the following effects of a rise in μ on k and m

$$\frac{dk}{d\mu} = \frac{(\xi + \lambda\delta_k)k - \lambda f}{\xi^2(\chi_1 f' - \chi_2)},$$

$$\frac{dm}{d\mu} = \left[(f' + \delta)\frac{dk}{d\mu} - \frac{m}{\xi}\right]\frac{\xi}{\mu_0} = \left[(f' + \delta)\frac{(\xi + \lambda\delta_k)k - \lambda f}{\xi^2(\chi_1 f' - \chi_2)} - \frac{f + \delta k}{\mu_0}\right]\frac{\xi}{\mu_0}. \tag{4.1.16}$$

As

$$\chi_1 f' - \chi_2 = \chi_1\left(f' - \frac{\chi_2}{\chi_1}\right) = \chi_1\left(f' - \frac{f}{k}\right) < 0,$$

$$(\xi + \lambda\delta_k)k - \lambda f = \left[(\xi + \lambda\delta_k) - \frac{\lambda\chi_2}{\chi_1}\right]k = \frac{-\xi^2 k}{\lambda\mu_0 - \xi^2} < 0,$$

we conclude $dk/d\mu > 0$. The sign of $dm/d\mu$ is the same as that of $(f' + \delta)\, dk/d\mu - m/\xi$. If $dk/d\mu$ is small and m is large, then the real money balance is reduced. In general, the impact is ambiguous. From $m = \chi c$ and equations (4.1.1), we have

$$\frac{dc}{d\mu} = \frac{1}{\chi}\frac{dm}{d\mu},$$

$$\frac{dr}{d\mu} = f''\frac{dk}{d\mu} < 0, \quad \frac{dw}{d\mu} = -kf''\frac{dk}{d\mu} > 0.$$

The rate of interest falls and the real wage rate rises as μ is increased. The impact on consumption is ambiguous. From $\bar{y} = f - \delta_k k + a$ and $a = \lambda\bar{y}$, we obtain

$$\frac{da}{d\mu} = (f' - \delta_k)\frac{\lambda}{\xi}\frac{dk}{d\mu}.$$

The total real wealth is increased if $f' > \delta_k$.

A change in χ has the following effects on k and m

$$\frac{dk}{d\chi} = \frac{\lambda f - (\xi + \lambda\delta_k)k}{\chi^2\xi^2(\chi_1 f' - \chi_2)} < 0,$$

$$\frac{dm}{d\chi} = \left[\left(f' + \delta\right)\frac{dk}{d\chi} + \frac{m}{\xi}\right]\frac{\xi}{\mu_0}. \tag{4.1.17}$$

An increase in χ reduces k. The impact on m is ambiguous. From $m = \chi c$ and equations (4.1.1), we have

$$\frac{dc}{d\chi} = \frac{1}{\chi}\frac{dm}{d\chi} - \frac{m}{\chi^2},$$

$$\frac{dr}{d\chi} = f''\frac{dk}{d\chi} > 0, \quad \frac{dw}{d\chi} = -kf''\frac{dk}{d\chi} < 0. \tag{4.1.18}$$

The rate of interest rises and the real wage rate falls as χ is increased. The impact on consumption is ambiguous. From $\bar{y} = f - \delta_k k + a$ and $a = \lambda\bar{y}$, we obtain

$$\frac{da}{d\chi} = \left(f' - \delta_k\right)\frac{\lambda}{\xi}\frac{dk}{d\chi}.$$

The total real wealth is increased if $f' > \delta_k$.

4.2 A CIA two-sector model

Huo (1977) examines a dynamic two-sector model in which money is introduced through a cash-in-advance constraint. A perfectly anticipated inflation changes the relative price between consumption goods, as they are differentially subject to the cash-in-advance constraint. The model foresees the inflation may either increase or decrease capital accumulation. The result is contrary to the conclusion of Stockman (1981). This section introduces cash-in-advance into Zhang's two-sector growth model with capital.[6]

4.2.1 The CIA two-sector monetary growth model

Like in the traditional two-sector growth model, it is assumed that consumption and capital goods are different commodities, which are produced in two distinct sectors. Labor is homogeneous and grows at an exogenously given exponential rate, n, which is assumed to be zero in this section. The assumption of zero population growth rate does not affect our analysis in the sense that we can get similar results with a constant growth rate (because the two sectors exhibit constant returns to scale). Capital depreciates at a constant exponential rate, δ_k,

which is independent of the manner of use. The production functions are given by

$$F_j\left(K_j(t),\, N_j(t)\right), \quad j = i, s,$$

where F_j are the output of sector j, K_j, and N_j are respectively the capital and labor used in sector j, F_j the production functions, the subscripts i and s denote the capital good sector and the consumption good sector. Assume F_j to be neoclassical. We have

$$f_j(t) = f_j\left(k_j(t)\right), \quad f_j(t) \equiv \frac{F_j(t)}{N_j(t)}, \quad k_j(t) \equiv \frac{K_j(t)}{N_j(t)}, \quad j = i, s.$$

Markets are competitive; thus labor and capital earn their marginal products, and firms earn zero profits. We introduce

N: fixed population;
$K(t)$: total capital stocks of the economy at time t;
$c(t)$: the level of per capita consumption of consumption goods;
$P_j(t)$: price of goods j, $j = i, s$, at time t; and
$r(t)$ and $w(t)$: the rate of interest and wage rate in terms of capital goods.[7]

The marginal conditions for the two sectors are

$$\begin{aligned} &r(t) + \delta_k = f_i'(k_i(t)) = p(t)\, f_s'(k_s(t)), \\ &w(t) = f_i(k_i(t)) - k_i(t)\, f_i'(k_i(t)) = p(t)\left[f_s(k_s(t)) - k_s(t)\, f_s'(k_s(t))\right], \end{aligned} \tag{4.2.1}$$

where $p(t) \equiv P_s(t)/P_i(t)$.

The total capital stock is allocated to the two sectors. As full employment of labor and capital is assumed, we have

$$K_i(t) + K_s(t) = K(t), \quad N_i(t) + N_s(t) = N.$$

We rewrite the above equations as

$$n_i(t) k_i(t) + n_s(t) k_s(t) = k(t), \quad n_i(t) + n_s(t) = 1, \tag{4.2.2}$$

where

$$k_j(t) \equiv \frac{K_j(t)}{N_j(t)}, \quad n_j(t) \equiv \frac{N_j(t)}{N}, \quad j = i, s.$$

Money is introduced by assuming that a central bank distributes at no cost to the population a per capita amount of fiat money $M(t) > 0$. The scheme according to

which the money stock evolves over time is deterministic and known to all agents. With μ being the constant net growth rate of the money stock, $M(t)$ evolves over time according: $\dot{M}(t) = \mu M(t), \mu > 0$.

It is assumed that the government uses $g(t)$ units of the output good per capita for public purpose

$$g(t) = \frac{\dot{M}(t)}{P_i(t)} = \frac{\mu M(t)}{P_i(t)} = \mu m(t), \tag{4.2.3}$$

where $m(t) \equiv M(t)/P_i(t)$.

When deciding about the composition of their portfolios, the household knows in advance that a certain fraction of consumption needs to be financed by payment in cash. Assume that cash has to be held in advance of purchasing goods. The liquidity constraint of the household is formed as[8]

$$M(t) \geq \chi P_s(t) c(t), \tag{4.2.4}$$

where χ is a positive parameter. We require $0 < \chi \leq 1$.[9]

From $M = mP_i$, we have

$$\pi(t) = \frac{\dot{P}_i(t)}{P_i(t)} = \mu - \frac{\dot{m}(t)}{m(t)}. \tag{4.2.5}$$

The disposable income is given by

$$\hat{y}(t) = y(t) + a(t) - \pi(t) m(t), \tag{4.2.6}$$

where

$$y(t) \equiv r(t) k(t) + w(t), \quad a(t) \equiv k(t) + m(t).$$

The budget constraint is given by

$$p(t) c(t) + s(t) = \hat{y}(t).$$

Inserting equation (4.2.6) into the budget constraint, we have

$$(1 + \chi\pi(t)) p(t) c(t) + s(t) = \bar{y}(t) = r(t) k(t) + w(t) + a(t), \tag{4.2.7}$$

where we use $m(t) = \chi p(t) c(t)$.

We specify the utility function in the Cobb-Douglas form

$$U(t) = c^{\xi_0}(t) s^{\lambda_0}(t), \quad \xi_0, \lambda_0 > 0.$$

The optimal solution is

$$c(t) = \frac{\xi \bar{y}(t)}{(1 + \chi\pi(t)) p(t)}, \quad s(t) = \lambda \bar{y}(t), \tag{4.2.8}$$

where

$$\xi \equiv \frac{\xi_0}{\xi_0 + \lambda_0}, \quad \lambda \equiv \frac{\lambda_0}{\xi_0 + \lambda_0}.$$

The wealth accumulation is given by

$$\dot{a}(t) = s(t) - a(t). \tag{4.2.9}$$

The balance of demand and supply in consumption goods market is given by

$$c(t)N = F_s(t). \tag{4.2.10}$$

We have thus built the economic system.

4.2.2 The dynamics and steady state

We show that the motion of the system is determined by a differential equation. First, from

$$r(t) + \delta_k = f_i'(k_i) = p f_s'(k_s)$$

in (4.2.1), we get

$$p = \frac{f_i'(k_i)}{f_s'(k_s)}.$$

Substituting this equation into $f_i(k_i) - k_i f_i'(k_i) = p\left(f_s(k_s) - k_s f_s'(k_s)\right)$ in equations (4.2.1) yields

$$\Psi_i(k_i) \equiv \frac{f_i(k_i)}{f_i'(k_i)} - k_i = \frac{f_s(k_s)}{f_s'(k_s)} - k_s \equiv \Psi_s(k_s). \tag{4.2.11}$$

From the properties of $f_i(k_i)$, we can show that the function, $\Psi_i(k_i)$, has the following properties

$$\Psi_i(0) = 0, \ \Psi_i(k_i) > 0 \quad \text{for } k_i > 0, \quad \Psi_i'(k_i) = -\frac{f_i(k_i) f_i''(k_i)}{f_i'^2(k_i)} \geq 0.$$

The function $\Psi_s(k_s)$ has the same properties in k_s. We see that for any given $k_s \geq 0$, equation (4.2.11) determines $k_i \geq 0$ as a unique function of k_s, denoted by

$$k_i = \Omega(k_s).$$

We have $\Omega(0) = 0$ and $\Omega' > 0$. From equations (4.2.2), we solve the labor distribution as functions of k, k_s, and k_i as follows

$$n_i = \frac{k - k_s}{k_i - k_s}, \quad n_s = \frac{k_i - k}{k_i - k_s}, \tag{4.2.12}$$

where we omit time expression.

From equation (4.2.10), we have

$$c = n_s f_s(k_s).$$

From this equation and equation (4.2.8), we have

$$\bar{y} = (1+\chi\pi)(k_i - k)\Omega_0(k_i), \tag{4.2.13}$$

where we also use equations (4.2.12), $p = f_i'(k_i)/f_s'(k_s)$ and

$$\Omega_0(k_i) \equiv \frac{f_i'(k_i) f_s(k_s)}{(k_i - k_s)\xi f_s'(k_s)}.$$

From the definition of $\bar{y}$ and a, we have

$$\bar{y} = (1+r)k + w + m. \tag{4.2.14}$$

Substituting equation (4.2.14) into equation (4.2.13), we solve

$$k = \frac{(1+\chi\pi)k_i\Omega_0(k_i) - w - m}{1+r+(1+\chi\pi)\Omega_0(k_i)}. \tag{4.2.15}$$

From $m(t) = \chi p(t)c(t)$ and (4.2.8), we solve

$$m = \frac{\xi\bar{y}}{1+\chi\pi}.$$

From this equation and equations (4.2.14) and (4.2.15), we get

$$(1+\chi\pi)m = (1+r)\xi\frac{(1+\chi\pi)k_i\Omega_0 - w - m}{(1+r)+(1+\chi\pi)\Omega_0} + \xi(w+m).$$

From this equation, we solve π as a function of k_i and m

$$\pi = \Omega_\pi(k_i, m) \equiv \frac{(1+r)\xi k_i\Omega_0 + (w+m)\xi\Omega_0 - m(1+r)}{m\chi\Omega_0} - \frac{1}{\chi}. \tag{4.2.16}$$

Substituting equation (4.2.16) into equation (4.2.15) yields

$$k = \Omega_k(k_i, m) \equiv k_i - \frac{m}{\xi\Omega_0(k_i)}. \tag{4.2.17}$$

From equations (4.2.5) and (4.2.16), we have

$$\dot{m} = \tilde{\Omega}_m(k_i, m) \equiv \mu m - m\Omega_\pi(k_i, m). \tag{4.2.18}$$

We see that the motion of $m(t)$ can be expressed as a function of $k_i(t)$ and $m(t)$.

Substituting $s = \lambda\bar{y}$ and $a = k + m$ into equation (4.2.9), we have

$$\dot{k} = \lambda w - (\xi - \lambda r)k - \xi m - \tilde{\Omega}_m(k_i, m), \tag{4.2.19}$$

where we use equation (4.2.14). Taking derivatives of equation (4.2.17) with respect to t yields

$$\dot{k} = \frac{\partial \Omega_k}{\partial k_i}\dot{k}_i + \frac{\partial \Omega_k}{\partial m}\tilde{\Omega}_m(k_i, m),$$

where

$$\frac{\partial \Omega_k}{\partial k_i} = 1 + \frac{m}{\xi \Omega_0^2(k_i)}\frac{\partial \Omega_0}{\partial k_i}, \quad \frac{\partial \Omega_m}{\partial k_i} = -\frac{1}{\xi \Omega_0(k_i)}.$$

Substituting the above equation and equation (4.2.17) into (4.2.19) yields

$$\dot{k}_i = \tilde{\Omega}_k(k_i, m) \equiv \left[\lambda w - (\xi - \lambda r)\Omega_k - \xi m - \tilde{\Omega}_m - \frac{\partial \Omega_k}{\partial m}\tilde{\Omega}_m\right]\left(\frac{\partial \Omega_k}{\partial k_i}\right)^{-1}. \tag{4.2.20}$$

We summarize the analytical results in the following lemma.

Lemma 4.2.1

The motion of the economic system is described by two differential equations with $k_i(t)$ and $m(t)$ as variables

$$\dot{k}_i = \tilde{\Omega}_k(k_i, m),$$
$$\dot{m} = \tilde{\Omega}_m(k_i, m). \tag{4.2.21}$$

All the other variables are determined as functions of $k_i(t)$ and $m(t)$ by the following procedure: $k(t)$ by (4.2.17) $\rightarrow \pi$ by (4.2.16) $\rightarrow a = k + m \rightarrow w$ and r by (4.2.1) $\rightarrow k_s$ by (4.2.11) $\rightarrow f_j(k_j)$, $j = i, s \rightarrow n_j \rightarrow p = f_i'(k_i)/f_s'(k_s) \rightarrow N_j = n_j N \rightarrow F_j = f_j(k_j)N_j \rightarrow c$ by (4.2.10) $\rightarrow \bar{y}$ by (4.2.13) $\rightarrow s = \lambda\bar{y}$.

From equations (4.2.18) and (4.2.20), at steady state we have[10]

$$\mu - \Omega_\pi(k_i, m) = 0,$$
$$\lambda w - (\xi - \lambda r)\Omega_k - \xi m = 0.$$

Substituting the definitions of $\Omega_\pi(k_i, m)$ and $\Omega_k(k_i, m)$ into the above equations, we solve

$$m = \frac{(1+r)k_i + w}{(\lambda + \chi\mu)\Omega_0 + (1+r)}\xi\Omega_0,$$
$$\lambda w - (\xi - \lambda r)k_i + \frac{(\xi - \lambda r)m}{\xi\Omega_0(k_i)} - \xi m = 0. \tag{4.2.22}$$

Substituting the first equation in (4.2.22) into the second one, we have

$$\lambda w-(\xi-\lambda r)k_i+\frac{\left(\xi-\lambda r-\xi^2\Omega_0\right)\left((1+r)k_i+w\right)}{(\lambda+\chi\mu)\Omega_0+(1+r)}=0. \tag{4.2.23}$$

For convenience of discussing equation (4.2.23), we specify $f_j=A_jk_j^{\alpha_j}$, $0<\alpha_j<1$. It is straightforward to confirm

$$k_s=\alpha k_i,\quad p=\frac{\beta_iA_i}{\beta_s\alpha^{\alpha_s}A_s}k_i^{\alpha_i-\alpha_s},\quad \Omega_0(k_i)=\alpha_1k_i^{-\beta_i}, \tag{4.2.24}$$

where

$$\alpha_1\equiv\frac{\alpha\alpha_iA_i}{(1-\alpha)\xi\alpha_s},\quad \alpha\equiv\frac{\beta_i\alpha_s}{\beta_s\alpha_i},\quad \beta_j\equiv1-\alpha_j,\quad j=i,s.$$

The price of consumer goods is positively related to the technological level of the capital good sector but negatively related to that of the consumer good sector. The price is positively or negatively related to the capital intensity of the capital good sector, depending on the sign of $\alpha_i-\alpha_s$. If $\alpha_i=\alpha_s$, the price is constant, $p=A_i/A_s$. In the remainder of this section, we require $\alpha_i\neq\alpha_s$. If the equality holds, labor distribution is invariant in time. The analysis of case $\alpha_i=\alpha_s$ is straightforward. We solve the capital intensity as follows

$$k_i=\left[\frac{-A_1\pm\sqrt{A_1^2-4A_2}}{2}\right]^{1/\beta_i}, \tag{4.2.25}$$

where we use $w=\beta_iA_ik_i^{\alpha_i}$, $r=\alpha_iA_ik_i^{-\beta_i}-\delta_k$, and

$$\delta\equiv1-\delta_k>0,\ \delta_\xi\equiv\xi+\lambda\delta_k>0,\quad A_1\equiv\frac{\beta_iA_i-\left(\lambda\delta_\xi+\chi\mu\delta_\xi+\delta\xi^2\right)\alpha_1}{2\delta\delta_\xi},$$

$$A_2\equiv(\lambda\chi\mu+\lambda-\xi)\frac{\alpha_1A_i}{2\delta\delta_\xi}.$$

As we should require $k_i>0$, we see that if $A_2<0$, then equation (4.2.23) has a unique solution. Since the sign of A_2 is the same as that of the term

$$\frac{\lambda\chi\mu+\lambda-\xi}{1-\alpha}=\frac{(\lambda\chi\mu+\lambda-\xi)\beta_s\alpha_i}{\alpha_i-\alpha_s}$$

we see that if λ is larger than ξ and $\alpha_s>\alpha_i$, then $A_2<0$. It should be remarked that even when A_2 is positive, we may also have a unique solution.

Lemma 4.2.2

Let the production functions take on the Cobb-Douglas form, if $\lambda_0>\xi_0$ and $\alpha_s>\alpha_i$, the dynamic system has a unique steady state.

4.2.3 The inflation policy dynamics and steady state

This section examines effects of change in the inflation policy on the economic system. We conduct the comparative statics analysis under the conditions of $\lambda_0 > \xi_0$ and $\alpha_s > \alpha_i$. The system has a unique equilibrium point. Taking derivatives of equation (4.2.25) with regard to μ yields

$$\frac{dk_i}{d\mu} = \left(\frac{\lambda A_i}{\alpha_1 \delta_\xi} - k_i^{\beta_i}\right) \frac{\chi \alpha_1^2 k_i}{2\delta\beta_i \left(2k_i^{2\beta_i} + A_1 k_i^{\beta_i}\right)}. \tag{4.2.26}$$

We note

$$2k_i^{2\beta_i} + A_1 k_i^{\beta_i} = k_i^{2\beta_i} - A_2 > 0.$$

The sign of $dk_i/d\mu$ is the same as that of $\lambda A_i/\delta_\xi - \alpha_1 k_i^{\beta_i}$. As

$$\frac{\lambda A_i}{\alpha_1 \delta_\xi} - k_i^{\beta_i} = \frac{\lambda A_i}{\alpha_1 \delta_\xi} + \frac{A_1}{2} - \frac{\sqrt{A_1^2 - 4A_2}}{2},$$

where $\lambda A_i/\alpha_1 \delta_\xi > 0, \sqrt{A_1^2 - 4A_2} > 0$, and $A_1/2$ may be positive or negative, we see that the impact is ambiguous. From

$$\frac{\lambda A_i}{\alpha_1 \delta_\xi} = \frac{(\alpha_i - \alpha_s)\xi}{(\xi/\lambda + \delta_k)\beta_i \alpha_i}$$

and

$$A_1 = \left[1 - \frac{\alpha_i \left(\lambda \delta_\xi + \chi \mu \delta_\xi + \delta \xi^2\right)}{(\alpha_i - \alpha_s)\xi}\right] \frac{\beta_i A_i}{2\delta \delta_\xi},$$

we see that if $\alpha_i - \alpha_s$ is positive and sufficiently small, then we have

$$\frac{\lambda A_i}{\alpha_1 \delta_\xi} + \frac{A_1}{2} < 0,$$

which guarantees $dk_i/d\mu < 0$. In the reminder of this section, we analyze the effects under $dk_i/d\mu < 0$. From $w = \beta_i A_i k_i^{\alpha_i}$ and $r = \alpha_i A_i k_i^{-\beta_i} - \delta_k$, we have

$$\frac{dw}{d\mu} = \frac{\alpha_i w}{k_i} \frac{dk_i}{d\mu} < 0, \quad \frac{dr}{d\mu} = -\frac{\beta_i A_i}{k_i^{1+\beta_i}} \frac{dk_i}{d\mu} > 0.$$

The real wage rate falls and the rate of interest rises. From equations (4.2.24), we have

$$\frac{dk_s}{d\mu} = \alpha \frac{dk_i}{d\mu} < 0, \quad \frac{dp}{d\mu} = \frac{(\alpha_i - \alpha_s)p}{k_i} \frac{dk_i}{d\mu} < 0.$$

Substituting the first equation in (4.2.22) into the second one, we have

$$\lambda w-(\xi-\lambda r)k_i+\frac{\left(\xi-\lambda r-\xi^2\Omega_0\right)\left((1+r)k_i+w\right)}{(\lambda+\chi\mu)\Omega_0+(1+r)}=0. \tag{4.2.23}$$

For convenience of discussing equation (4.2.23), we specify $f_j = A_j k_j^{\alpha_j}$, $0<\alpha_j<1$. It is straightforward to confirm

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$$k_i=\left[\frac{-A_1\pm\sqrt{A_1^2-4A_2}}{2}\right]^{1/\beta_i}, \tag{4.2.25}$$

where we use $w=\beta_i A_i k_i^{\alpha_i}$, $r=\alpha_i A_i k_i^{-\beta_i}-\delta_k$, and

$$\delta\equiv 1-\delta_k>0,\ \delta_\xi\equiv\xi+\lambda\delta_k>0,\quad A_1\equiv\frac{\beta_i A_i-\left(\lambda\delta_\xi+\chi\mu\delta_\xi+\delta\xi^2\right)\alpha_1}{2\delta\delta_\xi},$$

$$A_2\equiv(\lambda\chi\mu+\lambda-\xi)\frac{\alpha_1 A_i}{2\delta\delta_\xi}.$$

As we should require $k_i>0$, we see that if $A_2<0$, then equation (4.2.23) has a unique solution. Since the sign of A_2 is the same as that of the term

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4.2.3 The inflation policy dynamics and steady state

This section examines effects of change in the inflation policy on the economic system. We conduct the comparative statics analysis under the conditions of $\lambda_0 > \xi_0$ and $\alpha_s > \alpha_i$. The system has a unique equilibrium point. Taking derivatives of equation (4.2.25) with regard to μ yields

$$\frac{dk_i}{d\mu} = \left(\frac{\lambda A_i}{\alpha_1 \delta_\xi} - k_i^{\beta_i}\right) \frac{\chi \alpha_1^2 k_i}{2\delta\beta_i \left(2k_i^{2\beta_i} + A_1 k_i^{\beta_i}\right)}. \tag{4.2.26}$$

We note

$$2k_i^{2\beta_i} + A_1 k_i^{\beta_i} = k_i^{2\beta_i} - A_2 > 0.$$

The sign of $dk_i/d\mu$ is the same as that of $\lambda A_i/\delta_\xi - \alpha_1 k_i^{\beta_i}$. As

$$\frac{\lambda A_i}{\alpha_1 \delta_\xi} - k_i^{\beta_i} = \frac{\lambda A_i}{\alpha_1 \delta_\xi} + \frac{A_1}{2} - \frac{\sqrt{A_1^2 - 4A_2}}{2},$$

where $\lambda A_i/\alpha_1\delta_\xi > 0, \sqrt{A_1^2 - 4A_2} > 0$, and $A_1/2$ may be positive or negative, we see that the impact is ambiguous. From

$$\frac{\lambda A_i}{\alpha_1 \delta_\xi} = \frac{(\alpha_i - \alpha_s)\xi}{(\xi/\lambda + \delta_k)\beta_i \alpha_i}$$

and

$$A_1 = \left[1 - \frac{\alpha_i\left(\lambda\delta_\xi + \chi\mu\delta_\xi + \delta\xi^2\right)}{(\alpha_i - \alpha_s)\xi}\right] \frac{\beta_i A_i}{2\delta\delta_\xi},$$

we see that if $\alpha_i - \alpha_s$ is positive and sufficiently small, then we have

$$\frac{\lambda A_i}{\alpha_1 \delta_\xi} + \frac{A_1}{2} < 0,$$

which guarantees $dk_i/d\mu < 0$. In the reminder of this section, we analyze the effects under $dk_i/d\mu < 0$. From $w = \beta_i A_i k_i^{\alpha_i}$ and $r = \alpha_i A_i k_i^{-\beta_i} - \delta_k$, we have

$$\frac{dw}{d\mu} = \frac{\alpha_i w}{k_i}\frac{dk_i}{d\mu} < 0, \quad \frac{dr}{d\mu} = -\frac{\beta_i A_i}{k_i^{1+\beta_i}}\frac{dk_i}{d\mu} > 0.$$

The real wage rate falls and the rate of interest rises. From equations (4.2.24), we have

$$\frac{dk_s}{d\mu} = \alpha\frac{dk_i}{d\mu} < 0, \quad \frac{dp}{d\mu} = \frac{(\alpha_i - \alpha_s)p}{k_i}\frac{dk_i}{d\mu} < 0.$$

The capital intensity of the service sector and the price of services fall. From $\Omega_0 = \alpha_1 k_i^{-\beta_i}, \pi = \mu$ and equation (4.2.16), we have

$$\frac{dm}{d\mu} = \left[\left(\frac{(1+r)\xi}{m} - \frac{\delta\beta_i k_i^{-\alpha_i}}{\alpha_1}\right)\frac{dk_i}{d\mu} - \chi\right]\frac{m^2}{(1+r)\xi k_i + \xi w}.$$

As

$$\frac{(1+r)\xi}{m} - \frac{\delta\beta_i k_i^{-\alpha_i}}{\alpha_1} = \left[\left[\frac{(\lambda+\chi\mu)(1+r)\alpha_i}{(\alpha_i-\alpha_s)\xi} - \delta\beta_i\right]\beta_i A_i k_i^{-\beta_i} + (1+r-\delta\beta_i)(1+r)\right]\left[(1+r)k_i + w\right]\Omega_0,$$

we see that $\alpha_i - \alpha_s$ is positive and sufficiently small, then the above term is positive. Hence, we have $dm/d\mu < 0$. The real money balance is reduced due to the increase in the inflation rate. From equation (4.2.17), we have

$$\frac{dk}{d\mu} = \left(\frac{\alpha_i k_i + \beta_i k}{k_i}\right)\frac{dk_i}{d\mu} + \frac{k_i^{\beta_i}}{\xi\alpha_1}\frac{dm}{d\mu} < 0.$$

We see that the per capita physical wealth falls. As $a = k + m$, the per capita wealth falls as the inflation rate is increased. From equations (4.2.12) and $k_s = \alpha k_i$, we have

$$\frac{dn_s}{d\mu} = \left[(k-k_i)\alpha_i\frac{dk_i}{d\mu} - \frac{k_i^{1+\beta_i}}{\xi\alpha_1}\frac{dm}{d\mu}\right]\frac{k}{(1-\alpha)k_i^2}.$$

As $k_s = \alpha k_i$ and

$$1 - \alpha = \frac{\alpha_i - \alpha_s}{\beta_s\alpha_i} > 0,$$

we have $k_s < k_i$. As $k_i - k$ and $k_i - k_s$ have the same sign, we have $k - k_i < 0$. We have $dn_s/d\mu > 0$. That is, some workers changes their jobs in the industrial sector to the service sector.

From $\bar{y} = (1+\chi\mu)m/\xi$, we have

$$\frac{d\bar{y}}{d\mu} = \frac{\chi m}{\xi} + \left(\frac{1+\chi\mu}{\xi}\right)\frac{dm}{d\mu}.$$

We see that the impact on $\bar{y}$ is ambiguous. The impact on s has the same sign as that of $d\bar{y}/d\mu$. From $m = \chi pc$, we have

$$\frac{1}{c}\frac{dc}{d\mu} = \frac{1}{m}\frac{dm}{d\mu} - \frac{1}{p}\frac{dp}{d\mu}.$$

As m and p fall, the impact on the consumption level is ambiguous.

4.3 Monetary policy in a CIA economy

This section is concerned with an economy in which money is used to produce transaction services that are required to purchase consumption goods. For simplicity, we assume that there is a single consumption good that can be bought only for money (there are no credit cards).[11] There is a single input in the economy and the population is normalized to one. The household is endowed with 1 unit of time and choose endogenously to have leisure and work. We use a linear production function $f(t) = T(t)$, where $T(t)$ is the work time. The real wage income, $w(t)$, is given by $w(t) = T(t)$. Assume that cash has to be held in advance of purchasing goods. The liquidity constraint of the household is formed as

$$M(t) \geq \chi P(t) c(t), \tag{4.3.1}$$

where $M(t)$ is nominal money balance held for purchasing, $P(t)$ is the nominal price, and χ is a positive parameter.

Let $B(t)$ be the nominal bonds held by the household and $R_b(t) > 0$ be the nominal interest rate. The disposable income in nominal terms, $\hat{y}_m(t)$, is given by

$$\hat{y}_m(t) = P(t)T(t) + a_m(t) + R_b(t)B(t) - \pi(t)M(t) - \pi(t)B(t) - P(t)\tau(t),$$

where $\tau(t)(\geq 0)$ is the real lump tax, $\pi(t)$ is inflation rate, and $a_m(t)$ is the wealth in nominal terms defined as

$$a_m(t) \equiv M(t) + B(t).$$

The disposable income in real terms is given by

$$\hat{y}(t) = \frac{\hat{y}_m(t)}{P(t)} = T(t) + a(t) + R_b(t)b(t) - \pi(t)m(t) - \pi(t)b(t) - \tau(t), \tag{4.3.2}$$

where

$$a(t) \equiv \frac{M(t) + B(t)}{P(t)} = m(t) + b(t), \quad m(t) \equiv \frac{M(t)}{P(t)}, \quad b(t) \equiv \frac{B(t)}{P(t)}.$$

When the representative agent holds $m(t)$ units of real balances he loses $\pi(t)m(t)$ per unit of time, where $\pi(t) = \dot{P}(t)/P(t)$. The purchasing power of per unit of money is $1/P(t)$. The rate of change in the purchasing power is

$$\frac{1}{1/P(t)} \frac{d1/P(t)}{dt} = -\frac{\dot{P}(t)}{P(t)} = -\pi(t).$$

The budget constraint is given by

$$c(t) + s(t) = \hat{y}(t).$$

Inserting $T(t) = 1 - T_h(t)$ into the definition of $\hat{y}(t)$ and then substituting the resulted equation into $c(t)+s(t)=\hat{y}(t)$, we have

$$c(t)+s(t)+T_h(t)=\bar{y}(t)\equiv 1+a(t)+R_b(t)b(t) \\ -\pi(t)m(t)-\pi(t)b(t)-\tau(t). \tag{4.3.3}$$

We assume that at each point of time consumers' preferences over leisure time, consumption and saving can be represented by the following utility function

$$U(t)=T_h^{\sigma_0}(t)c^{\xi_0}(t)s^{\lambda_0}(t), \quad \sigma_0,\xi_0,\lambda_0>0,$$

where σ_0 is called propensity to use leisure time, ξ_0 propensity to consume, and λ_0 propensity to own wealth. For simplicity, we assume these propensities to be fixed.[12] Consumers' problem is to choose current consumption, leisure, and savings in such a way that utility levels are maximized. Maximizing $U(t)$ subject to the budget constraints yields

$$T_h(t)=\sigma\bar{y}(t), \quad c(t)=\frac{\xi\bar{y}(t)}{1+\chi R_b(t)}, \quad s(t)=\lambda\bar{y}(t), \tag{4.3.4}$$

where

$$\sigma\equiv\rho\sigma_0, \quad \xi\equiv\rho\xi_0, \quad \lambda\equiv\rho\lambda_0, \quad \rho\equiv\frac{1}{\sigma_0+\xi_0+\lambda_0}.$$

These conditions say that the total leisure measured in wage is proportional to the disposable personal income with the proportional rate being equal to the propensity to enjoy leisure. Similarly, the optimal consumption and savings are proportional to the disposable personal income.

The variable, $a(t)$, changes as follows[13]

$$\dot{a}(t)=s(t)-a(t). \tag{4.3.5}$$

Much recent discussion of monetary policy is centered on a class of policies known as Taylor rules, which assume that the interest rate set by the central bank as an increasing function of the inflation rate.[14] Since the publication of Taylor's paper, effects of active interest feedback rules (which say that increases in inflation are associated with a more than one-for-one increase in the nominal interest rate) on economic dynamics have caused a great interest in monetary economics. Many central banks today use some interest rate as the policy instrument. It is reasonable to expect that effects vary, according to in which theoretical framework the issues are examined.[15] As shown in a monetary economic model in the MIU approach by Benhabib and Uribe (2002) active interest rate feedback rules can easily lead to different consequences. Following Benhabib *et al.* (2001a) we assume that the monetary policy takes the form of Taylor's interest rate

feedback rule whereby the nominal rate is set as an increasing function of inflation as follows

$$R_b(t) = \Lambda(\pi(t)), \tag{4.3.6}$$

where $\Lambda(\pi)$ is continuous, non-decreasing in inflation rate.[16]

The government purchases are assumed zero at all times. The budget constraint of the government is given by

$$\dot{B}(t) + \dot{M}(t) + P(t)\tau(t) = R_b(t)B(t). \tag{4.3.7}$$

From $M(t) + B(t) = a(t)P(t)$, we can rewrite constraint (4.3.7) as follows

$$\dot{a}(t) = R_b(t)b(t) - \pi(t)a(t) - \tau(t). \tag{4.3.8}$$

We have thus established the model. We now examine the behavior of the model.

4.3.1 Dynamics and steady state

By equations (4.3.5) and (4.3.7) we have

$$s(t) - a(t) = R_b(t)b(t) - \pi(t)a(t) - \tau(t).$$

Substituting $s(t) = \lambda\bar{y}(t)$ into the above equation, we have[17]

$$\bar{y}(t) = \frac{1}{\sigma + \xi},$$

in which we also use

$$b(t) = \frac{\bar{y}(t) - 1 - a(t) + \pi(t)a(t) + \tau(t)}{R_b(t)}. \tag{4.3.9}$$

which is obtained from equation (4.3.3). The potential disposable income, $\bar{y}$, is constant, being dependent only on the preference and independent of the government policy. From equation (4.3.5) and $s(t) = \lambda\bar{y}(t)$, we directly solve

$$a(t) = \left(a(0) - \frac{\lambda}{\sigma + \xi}\right)\exp(-t) + \frac{\lambda}{\sigma + \xi}. \tag{4.3.10}$$

Although the real wealth is dependent on its initial value in the short term, it is solely determined by the preference in the long term. The monetary policy has no impact on $a(t)$, $T(t)$, and $c(t)$.

From $c(t) = \xi\bar{y}/(1+\chi R_b(t))$ and $m(t) = \chi c(t)$, we solve

$$m(t) = \frac{\chi\xi\bar{y}}{1+\chi R_b(t)}. \tag{4.3.11}$$

From $a = b + m$, we solve

$$b(t) = a(t) - \frac{\chi\xi\bar{y}}{1+\chi R_b(t)}. \tag{4.3.12}$$

Substituting this equation into equation (4.3.9), we have

$$\Omega(\pi(t)) \equiv (R_b(\pi(t)) - \pi(t))a(t) - \frac{\chi\xi\bar{y}R_b(\pi(t))}{1+\chi R_b(\pi(t))} - \bar{\tau}(t) = 0, \tag{4.3.13}$$

in which

$$\bar{\tau}(t) \equiv \tau(t) - \left(a(0) - \frac{\lambda}{\sigma+\xi}\right)\exp(-t).$$

As $a(t)$ and $\tau(t)$ are known, equation (4.3.13) contains a single variable, $\pi(t)$. We now examine properties of this equation. For illustration, we are only concerned with steady states and require τ being constant. At steady state, equation (4.3.13) becomes

$$\Omega(\pi) = \left[\frac{1+(1-\xi)\chi R_b(\pi)}{1+\chi R_b(\pi(t))}\right]R_b(\pi) - \pi - \frac{\tau}{\bar{y}} = 0.$$

It is straightforward to check for different forms of R_b,[18] the equation may have either a unique or multiple steady states. For instance, if we specify the policy as

$$R_b = \theta_0 + \theta_1\pi^\theta, \quad \theta_0, \theta_1, \theta > 0,$$

then a steady state is determined by

$$\frac{R_b + (1-\xi)\chi R_b^2}{1+\chi R_b} - \left(\frac{R_b - \theta_0}{\theta_1}\right)^{1/\theta} - \frac{\tau}{\bar{y}} = 0.$$

In the case of $\theta = 1$, we have

$$R_b = \frac{\bar{\theta}_1 \pm \sqrt{\bar{\theta}_1^2 + 4\bar{\theta}_2}}{2},$$

where

$$\bar{\theta}_1 \equiv \left(\chi\bar{\theta}_2 + \frac{1-\theta_1}{[(1-\xi)\theta_1 - 1]\chi}\right),$$

$$\bar{\theta}_2 \equiv \left(\frac{\theta_1\tau/\bar{y} - \theta_0}{[(1-\xi)\theta_1 - 1]\chi}\right).$$

Even with a simple policy, $R_b = 0.01 + \theta_1 \pi$, it is possible for the problem to have either unique solution or two solutions.[19] If we specify $R_b = \theta_0 + \theta_1 \pi^{1/n}$, with n being a positive integer, the equation may have many solutions. As demonstrated in Benhabib and Uribe, the key factor for whether the system has a unique or multiple steady states is the properties of $R_b(\pi)$. By the way, basing on the models in Sections 4.1–4.3, it is straightforward to extend this model to include capital accumulation.[20]

4.4 Home production in the CIA-OSG model

Becker (1965) and Lancaster (1966, 1971) introduced the concept of household production function. Instead of receiving utility directly from goods purchased in the market, they assumed that consumers derive utility from the attributes possessed by these goods, and then only after some transformation is performed on these market goods. A typical example is that the consumers purchase raw food in the market but they derive utility from the consumption of the completed meal, which has been produced by combining the raw food with labor and time in some environment. In *Treatise on the Family*, Becker (1981) emphasized the importance of division of labor and gains from specification. Becker applies the concept of household production functions to describe the possibility of producing household commodities with varied inputs. Household commodities include nonmarket goods including children, prestige, envy, health, and pleasures of the senses. These goods are produced with market goods and labor time of household members as inputs. Some goods, such as living space, household heating and lighting, and shared automobile trips are jointly consumed and can be considered as local public goods which enter simultaneously into the utility functions of all family members. It is argued that the because of benefits of shared public goods, marriage yields a utility surplus over living separately. The well-being of children is treated as a household public good that enters the utility of both parents whether these parents are married or divorced.[21] Perceiving household as a firm and applying microeconomic theories about firm behavior to the household, Becker examined specification within the household, comparative advantage, returns to scale, factor substitution, human capital, and assortative mating. Each member of the family can use time either for household labor or market labor and the family can purchase market goods either to consume or to use them as inputs in household production function.

4.4.1 The CIA-OSG model with home production

The OSG model in this section views family as production units.[22] In addition to enjoyment, households also supply labor services, recreation, spiritual experiences, as well as conventional goods of the do-it-yourself variety. Household activities are conducted within houses as well as outside doors. These activities not only consume goods (like eat fruits and vegetables) but also utilize machines like housing, video, TV, cooking machines, cars, and the like. This section extends

the one-sector growth model proposed in Section 2.1 to endogenously determine capital distribution between industrial and housing production.

Except home production, most aspects of this model are the same as the OSG model. The parameters, δ_k, N, and the variables, $F(t), w(t), r(t)$, and $C(t)$ are defined as in the OSG model. Total capital, $K(t)$, is distributed between the industrial sector and home use. We denote $K_i(t)$ and $K_h(t)$ respectively the capital stocks employed by the industrial sector and households. The per capita values of the two variables are respectively $k_i(t)$ and $K_h(t)$. The production function is: $F = K_i^{\alpha} N^{\beta}$, where α and β are parameters. The marginal conditions are

$$r + \delta_k = \alpha f'(k_i), \quad w = \beta f(k_i), \tag{4.4.1}$$

where $f = k_i^{\alpha}$ and $k_i \equiv K_i/N$. We fix $N = 1$. We have

$$\dot{M}(t) = \mu M(t), \quad \mu > 0,$$

$$g(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t). \tag{4.4.2}$$

The liquidity constraint of the household is formed as

$$m(t) = \chi\left[c(t) + (r(t) + \delta_k) k_h(t)\right]. \tag{4.4.3}$$

We require $0 < \chi \leq 1$. From $M = mP$, we have

$$\pi(t) = \frac{\dot{P}(t)}{P(t)} = \mu - \frac{\dot{m}(t)}{m(t)}.$$

The disposable income is given by

$$\hat{y} = rk + w + a - \pi m, \tag{4.4.4}$$

where

$$a(t) \equiv k(t) + m(t), \quad k(t) = k_i(t) + k_h(t).$$

The budget constraint is given by

$$c + (r + \delta_k) k_h + s = \hat{y}.$$

Insert equation (4.4.4) in the budget constraint

$$(1 + \chi\pi)c + (1 + \chi\pi)(r + \delta_k) k_h + s = \bar{y} = rk + w + a, \tag{4.4.5}$$

where we use equation (4.4.3). We assume that $U(t)$ is dependent on consumption level, $c(t)$, housing, $k_h(t)$, and savings, $s(t)$, in the following way

$$U = c^{\xi_0} k_h^{\eta_0} s^{\lambda_0}, \quad \xi_0, \eta_0, \lambda_0 > 0.$$

Maximizing U subject to equation (4.4.5) yields

$$(1+\chi\pi)c=\xi\bar{y},\quad (1+\chi\pi)(r+\delta_k)k_h=\eta\bar{y},\quad s=\lambda\bar{y}, \tag{4.4.6}$$

where

$$\xi\equiv\rho\xi_0,\quad \eta\equiv\rho\eta_0,\quad \lambda\equiv\rho\lambda_0,\quad \rho\equiv\frac{1}{\xi_0+\eta_0+\lambda_0}.$$

The above equations mean that the housing consumption and consumption of the good are positively proportional to the net income and capital wealth, and the saving is positively proportional to the net income but negatively proportional to the wealth.

The wealth accumulation is given by

$$\dot{a}(t)=s(t)-a(t). \tag{4.4.7}$$

We have thus built the model.

4.4.2 The dynamics and steady state

We first show that the dynamics can be expressed as a two differential equations system. First, from $r+\delta_k=\alpha f'$ in equations (4.4.1) and $(1+\chi\pi)(r+\delta_k)k_h=\eta\bar{y}$, we obtain

$$k_h=\frac{\eta\bar{y}}{(1+\chi\pi)\alpha f'}. \tag{4.4.8}$$

We note that r and w are uniquely determined as functions of k_i. From $\bar{y}=(1+r)k+w+m$, we see that $\bar{y}$ is a function of k_i, k, and m. Substituting equation (4.4.2) into $k_h+k_i=k$ yields

$$k=\frac{\eta w+\eta m+(1+\chi\pi)\alpha f'k_i}{(1+\chi\pi)\alpha f'-(1+r)\eta}, \tag{4.4.9}$$

where we use $\bar{y}=(1+r)k+w+m$. Substituting equations (4.4.6) into equation (4.4.3) yields

$$\pi=\frac{(1+r)\bar{\lambda}k+\bar{\lambda}w}{m}+\bar{\chi}, \tag{4.4.10}$$

where we use $\bar{y}=(1+r)k+w+m$ and

$$\bar{\lambda}\equiv\xi+\eta,\quad \bar{\chi}\equiv\frac{(\xi+\eta)\chi-1}{\chi}.$$

Insert equation (4.4.9) in (4.4.10)

$$\pi^2 + f_1(m, k_i)\pi - f_2(m, k_i) = 0,$$

where

$$f_1(m, k_i) \equiv \frac{(\alpha f' - \eta - \eta r)m - (1+r)\bar{\lambda}\alpha\chi f' k_i - \left(\bar{\lambda}w + m\bar{\chi}\right)\chi\alpha f'}{\chi\alpha f' m},$$

$$f_2(m, k_i) \equiv \frac{(w+m)(1+r)\eta\bar{\lambda} + (1+r)\bar{\lambda}\alpha f' k_i + (\alpha f' - \eta - \eta r)\left(\bar{\lambda}w + m\bar{\chi}\right)}{\chi\alpha f' m}.$$

Hence, we have

$$\pi_{1,2} = \Lambda_{1,2}(m, k_i) \equiv \frac{-f_1(m, k_i) \pm \sqrt{f_1^2(m, k_i) + 4f_2(m, k_i)}}{2}. \tag{4.4.11}$$

From $\pi = \mu - \dot{m}/m$ and equation (4.4.11), we obtain

$$\dot{m} = \mu m - m\Lambda(m, k_i), \tag{4.4.12}$$

where Λ represents either of the two solutions, $\Lambda_{1,2}(m, k_i)$. From equations (4.4.9) and (4.4.11), we have

$$k = \bar{\Lambda}(m, k_i) \equiv \frac{\eta w + \eta m + (1 + \chi\Lambda(m, k_i))\alpha f' k_i}{(1 + \chi\Lambda(m, k_i))\alpha f' - (1+r)\eta}.$$

Hence, we can represent k as a function of m and k_i. From $s = \bar{\lambda}\bar{y}$ and $a = k + m$, we have

$$\dot{k} = (\lambda r - \xi)k + \lambda w - (\xi + \mu)m + m\Lambda(m, k_i), \tag{4.4.13}$$

where we use equation (4.4.12) and $\bar{y} = (1+r)k + w + m$. Taking derivatives of equation (4.4.13) with respect to t yields

$$\dot{k} = \bar{\Lambda}_m \dot{m} + \bar{\Lambda}_{k_i} \dot{k}_i,$$

where $\bar{\Lambda}_m$ and $\bar{\Lambda}_{k_i}$ are respectively partial derivatives of $\bar{\Lambda}$ with respect to m and k_i. Substituting equation (4.4.13) and the above equation into (4.4.13) yields

$$\dot{k}_i = \frac{(\lambda r - \xi)k + \lambda w - (\xi + \mu)m + m\Lambda - \mu m\bar{\Lambda}_m + m\bar{\Lambda}_m\Lambda}{\bar{\Lambda}_{k_i}}, \tag{4.4.14}$$

where we use equation (4.4.12).

Lemma 4.4.1

The motion of the economic system is described by two differential equations (4.4.12) and (4.4.14) with $k_i(t)$ and $m(t)$ as variables. All the other variables are determined as functions of $k_i(t)$ and $m(t)$ by the following procedure: k by (4.4.13) $\rightarrow k_h = k - k_i \rightarrow \pi$ by (4.4.11) $\rightarrow a = k + m \rightarrow w$ and r by (4.4.1) $\rightarrow f(k_i) \rightarrow F = f(k_i)N \rightarrow c$ by (4.2.10) $\rightarrow \bar{y} = (1+r)k + w + m \rightarrow c$ and s by equations (4.4.6).

We now examine steady states. From $\pi = \mu - \dot{m}/m$, we have $\pi = \mu$. The inflation rate is uniquely determined. By equation (4.4.7), we have $\lambda\bar{y} = a$. From $\lambda\bar{y} = a$ and $\bar{y} = rk + w + a$, we have

$$a = \frac{\lambda r k + \lambda w}{\xi}. \tag{4.4.15}$$

From equations (4.4.8), (4.4.15), and $\lambda\bar{y} = a$, we have

$$k_h = \frac{\eta r k + \eta w}{(1+\chi\mu)\alpha\xi f'}.$$

From this equation and $k_i + k_h = k$, we solve

$$k = \frac{\bar{\chi} w + k_i f'}{f' - \bar{\chi} r} = \frac{(\bar{\chi}\beta + \alpha)Ak_i}{\delta_k \bar{\chi} k_i^{\beta} - (\bar{\chi} - 1)\alpha A}, \tag{4.4.16}$$

where $\bar{\chi} \equiv \eta/\alpha\xi(1+\chi\mu)$ and use $r = \alpha A k_i^{-\beta} - \delta_k$ and $w = \beta A k_i^{\alpha}$. As $\chi\mu$ tends to be small, α is less than unit, if we require η to be larger than $\alpha\xi$,[23] we have $\bar{\chi} > 1$. In the remainder of this section, we require this inequality to hold. From $\lambda\bar{y} = a$, $\pi = \mu$ and equations (4.4.3) and (4.4.6), we have

$$(1+\chi\mu)m = (\xi + \eta)\chi a.$$

Insert $m = a - k$ and equation (4.4.15) in the above equation, we obtain

$$a = \bar{\mu} k,$$

where

$$\bar{\mu} \equiv \frac{1+\chi\mu}{1+\chi\mu - \xi\chi - \eta\chi} > 0,$$

where we use $\chi \leq 1$ and $\xi + \eta + \lambda = 1$ in judging the sign of $\bar{\mu}$. From $a = \bar{\mu}k$ and equation (4.4.15), we have

$$k = \frac{\lambda\beta A k_i}{(\xi\bar{\mu} + \lambda\delta_k)k_i^{\beta} - \lambda\alpha A}, \tag{4.4.17}$$

where we also use $r = \alpha A k_i^{-\beta} - \delta_k$ and $w = \beta A k_i^{\alpha}$. From this equation and equation (4.4.16), we have

$$\Omega(k_i) \equiv \frac{\lambda\beta}{(\xi\overline{\mu} + \lambda\delta_k)k_i^{\beta} - \lambda\alpha A} - \frac{\overline{\chi}\beta + \alpha}{\delta_k\overline{\chi}k_i^{\beta} - (\overline{\chi} - 1)\alpha A} = 0. \tag{4.4.18}$$

For a solution to be meaningful, it is necessary to require

$$(\xi\overline{\mu} + \lambda\delta_k)k_i^{\beta} - \lambda\alpha A > 0, \quad \delta_k\overline{\chi}k_i^{\beta} - (\overline{\chi} - 1)\alpha A > 0,$$

that is[24]

$$k_i > k_0 \equiv \max\left\{\left(\frac{\lambda\alpha A}{\xi\overline{\mu} + \lambda\delta_k}\right)^{1/\beta_i}, \left[\frac{(\overline{\chi} - 1)\alpha A}{\delta_k\overline{\chi}}\right]^{1/\beta_i}\right\}. \tag{4.4.19}$$

Equation (4.4.18) has a unique solution as follows

$$k_i = \left[\frac{\alpha\lambda A}{\overline{\chi}\beta\xi\overline{\mu} + \alpha(\xi\overline{\mu} + \lambda\delta_k)}\right]^{1/\beta}. \tag{4.4.20}$$

It can be seen that if

$$(1 - \overline{\chi})\beta\xi\overline{\mu} + \beta\lambda\delta_k > 0, \quad \frac{(\beta\lambda\delta_k\overline{\chi} + \alpha\lambda\delta_k)}{\overline{\chi} - 1} > (\overline{\chi}\beta + \alpha)\xi\overline{\mu}, \tag{4.4.21}$$

are satisfied, then k_i satisfies (4.4.18). As the above inequalities are satisfied if, for instance, χ is near unit. Hence, it is not strict to require (4.4.10). In summary, we get the following lemma.

Lemma 4.4.2

Let inequalities (4.4.20) be satisfied. Then the dynamic system has a unique steady state. The values of k_i and k are given by equations (4.4.19) and (4.4.16). All the other variables are determined as functions of k_i and k by the following procedure: $a = \overline{\mu}k \to m = a - k \to k_h = k - k_i \to \pi = \mu \to w$ and r by (4.4.1) $\to f(k_i) \to F = f(k_i)N \to c$ by (4.2.10) $\to \overline{y} = (1 + r)k + w + m \to c$ and s by equations (4.4.6).

Although it is straightforward to calculate the eigenvalues of the steady state, it is difficult to interpret the stability conditions because expressions are tedious. We omit the stability conditions.

4.4.3 The inflation and economic structure

We now examine effects of changes in μ. Taking derivatives of equation (4.4.19) with respect to

$$\frac{dk_i}{d\mu} = \frac{\xi\mu\overline{\chi}\beta\overline{\mu}k_i}{[\overline{\chi}\beta\xi\overline{\mu} + \alpha(\xi\overline{\mu} + \lambda\delta_k)](1 + \chi\mu)\beta} + \frac{(\overline{\chi}\beta + \alpha)(\xi + \eta)\chi k_i}{(1 + \chi\mu - \xi\chi - \eta\chi)^2} > 0.$$

The capital intensity of the production sector is increased as the inflation rate rises. We have

$$\frac{dr}{d\mu} = -\alpha\beta A k_i^{-\beta-1}\frac{dk_i}{d\mu} < 0, \quad \frac{dw}{d\mu} = \alpha\beta A k_i^{-\beta}\frac{dk_i}{d\mu} > 0, \quad \frac{df}{d\mu} = \frac{\alpha f}{k_i}\frac{dk_i}{d\mu} > 0.$$

The rate of interest rises and the wage rate falls. The per worker output level is also increased. From equation (4.4.17), we have

$$\frac{dk}{d\mu} = \left[-\frac{\alpha}{k_i^{1+\beta}}\frac{dk_i}{d\mu} + \frac{(\xi+\eta)\chi\xi}{(1+\chi\mu-\xi\chi-\eta\chi)^2\lambda A}\right]\frac{k^2}{k_i^{\alpha}\beta}.$$

As the expression is tedious, it is difficult to explicitly determine the effect on the total physical wealth. From $a = \overline{\mu}k$ and $m = (\overline{\mu}-1)k$, we obtain

$$\frac{da}{d\mu} = \frac{a}{k}\frac{dk}{d\mu} - \frac{(\xi+\eta)\chi k}{(1+\chi\mu-\xi\chi-\eta\chi)^2},$$
$$\frac{dm}{d\mu} = \frac{a}{m}\frac{dk}{d\mu} - \frac{(\xi+\eta)\chi m}{(1+\chi\mu-\xi\chi-\eta\chi)^2}.$$

If $dk/d\mu < 0$, then the total wealth and the real money balances fall; otherwise, the effects on the total wealth and the real money balances are ambiguous. From $\lambda\overline{y} = a, \pi = \mu$ and equations (4.4.6), we obtain

$$\frac{dc}{d\mu} = \frac{c}{a}\frac{da}{d\mu} - \frac{\chi c}{1+\chi\mu},$$
$$\frac{dk_h}{d\mu} = \frac{k_h}{a}\frac{da}{d\mu} - \frac{\chi k_h}{1+\chi\mu} + \frac{\beta k_h}{k_i}\frac{dk_i}{d\mu}.$$

If $dk/d\mu < 0$, per capita consumption level falls; otherwise, the impact on the per capita consumption level is ambiguous. Even when $dk/d\mu < 0$, the impact on housing consumption may be positive because as the rate of interest falls, the consumption rises.

4.5 Growth with learning by doing and research[25]

This section introduces knowledge accumulation via research into the monetary growth model in Section 4.1. Dynamics of occupational choice may influence economic development in different ways.[26] As distribution affects saving, investment, risk bearing, fertility, demand, and production, development processes are connected to distribution. On the other hand, development also affects the structure of occupations. It changes the demand for and supply of different types of labor. For instance, as an economy is industrializing, the labor share in industrial and service sectors tend to increase, while that of the agricultural sector tends to decline.

This section proposes a dynamic model of the economic system which consists of one production sector and one university.[27] There are two professions, to do science and to work in the production sector. The university is financially supported by the government. The production sector is the same as in the OSG model developed in Chapter 2. Demand for commodity, saving and labor distribution are endogenously determined. A typical household maximizes its utility, which is dependent on the wealth and the consumption level of commodity.[28] Knowledge growth is through learning by doing by the production sector and R&D activities by the university.

4.5.1 The monetary growth model with research

For convenience, we term the people working in the production sector as workers and the people working in the university as scientists. The population is classified into workers and scientists. Let subscript indexes i, and r stand for the production sector and the university, respectively. We define N, $K(t)$, $F(t)$, $w(t)$, $r(t)$, $k(t)$, $c(t)$, $s(t)$, $M(t)$, $P(t)$, $m(t)$ and δ_k as in Section 4.1. Let $N_j(t)$ and $K_j(t)$ stand for the labor force and the capital stocks employed by sector $j, j = i, r$.

The production sector

The production is a process of combining labor force, capital and knowledge. The production function is specified as follows

$$F(t) = AZ^{\vartheta}(t)K_i^{\alpha}(t)N_i^{\beta}(t), \quad A, \alpha, \beta > 0, \quad \vartheta \geq 0, \quad \alpha + \beta = 1, \tag{4.5.1}$$

in which $Z(t)$ is the knowledge stock of the economy and ϑ is the knowledge utilization efficiency parameter. Let τ stand for the (fixed) tax rate on the product level. Maximizing the profit by the production sector yields the following conditions

$$r + \delta_k = \frac{\alpha \bar{\tau} f(k_i)}{k_i}, \quad w = \beta \bar{\tau} Z^{\vartheta/\beta} f(k_i), \tag{4.5.2}$$

in which

$$\bar{\tau} \equiv 1 - \tau, \quad f(k_i) \equiv Ak_i^{\alpha}(t), \quad k_i(t) \equiv \frac{K_i(t)}{Z^{\vartheta/\beta}(t)N_i(t)}.$$

We assume that there is only one resource for the government to get its resource for supporting the university. There is no tax upon property or wage incomes. We may relax this assumption within the framework of this book by introducing heterogeneous tax rates upon production sectors and households' incomes from wages and properties.

As in Section 4.1, we have

$$\dot{M}(t) = \mu M(t), \quad \mu > 0,$$

$$g(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t). \tag{4.5.3}$$

$$M(t) \geq \chi P(t) c(t), \tag{4.5.4}$$

where χ is a positive parameter.[29] We require $0 < \chi < 1$. As in Section 4.1, the budget constraint is given by

$$(1 + \chi\pi(t))c(t) + s(t) = \bar{y}(t) = r(t)k(t) + w(t) + a(t), \tag{4.5.5}$$

where $a \equiv k + m$ and $\pi = \mu - \dot{m}/m$, and we use $m(t) = \chi c(t)$. With $U = c^{\xi_0} s^{\lambda_0}$, the optimal solution is

$$c(t) = \frac{\xi \bar{y}(t)}{1 + \chi\pi(t)}, \quad s(t) = \lambda \bar{y}(t), \tag{4.5.6}$$

where

$$\xi \equiv \frac{\xi_0}{\xi_0 + \lambda_0}, \quad \lambda \equiv \frac{\lambda_0}{\xi_0 + \lambda_0}.$$

The wealth accumulation is given by

$$\dot{a}(t) = s(t) - a(t). \tag{4.5.7}$$

We now describe behavior of the university and knowledge accumulation. Some major new knowledge and inventions that had far reaching and prolonged implications, such as Newton's mechanics, Einstein's theory of relativity, steam engine, electricity, and computer. Small improvements and non-lasting improvements take place everywhere, serendipitously and intentionally. Innovations may also happen in a drastic, discontinuous fashion, or in a slow, continuous manner. The introduction of the first steam engine rapidly triggered a sequence of innovations. The same is true about electricity and computer. In their seminal article on the theory of general purpose technologies, Bresnahan and Trajtenberg (1995) argued that technologies have a treelike structure, with a few prime movers located at the top and all other technologies radiating out from them. They characterize general purpose technologies by pervasiveness (which means that such a technology can be used in many downstream sectors), technological dynamism (which means that it can support continuous innovational efforts and learning), and innovational complementarities (which exist because productivity of R&D in downstream sectors increases as a consequence of innovation in the general purpose technology, and vice versa). A general purpose technology, such

as the computer, has the potential to effect the entire economic system and can lead to far-reaching changes in social factors such as working hours and constraints on family life.

We propose the following equation for knowledge growth

$$\dot{Z}(t) = \frac{\upsilon_i f(t)}{Z^{\vartheta_i}(t)} + \upsilon_r Z^{\vartheta_r}(t) K_r^{\alpha_r}(t) N_r^{\beta_r}(t) - \delta_z Z(t), \tag{4.5.8}$$

in which $\delta_z (\geq 0)$ is the depreciation rate of knowledge, and ϑ_j, υ_j, $(j = i, r)$, α_r and β_r are parameters. We require υ_j, α_r, and β_r to be non-negative. In equation (4.5.5), we omit possible R&D by private companies.

We have used the concept of knowledge, $Z(t)$, in a highly aggregated sense. Knowledge, like physical capital, is a complexity that has different forms. Knowledge ranges from the highly abstract to the highly applied. At one extreme is abstract knowledge with little applicability. We also have abstract knowledge with broad applicability, such as the general (restricted, special) theory of relativity and the theory of quantum mechanics. At another extreme, we have knowledge about special goods, such as how to write Japanese in a computer. Different types of knowledge play varied roles in economic growth. In order to fully understand these roles, we have to refine economic structures dividing a single production sector into multiple ones and classifying a homogeneous labor force into a heterogeneous one. Progress in any special type of knowledge may have different implications for different production sectors or different people.

The university is financially supported by the government. The university's budget is given by

$$(r(t) + \delta_k) K_r(t) + w_r(t) N_r(t) = G(t) \equiv \tau F(t) + \mu N m(t). \tag{4.5.9}$$

The university pays the rent, rK_r, for the equipments and the scientists' wage, $w_r N_r$, and obtains the research fund, $\tau F + \mu N m$, from the government. We now design the distribution policy to determine the number of scientists and the amount of equipments.

The university's problem is formed as follows

$$Max\, \upsilon_r Z^{\vartheta_r}(t) K_r^{\alpha_r}(t) N_r^{\beta_r}(t)$$

subject to equation (4.5.9). The solution is given by

$$K_r = \frac{\alpha_r G}{\alpha_0 (r + \delta_k)}, \quad N_r = \frac{\beta_r G}{\alpha_0 w}, \tag{4.5.10}$$

where $\alpha_0 \equiv \alpha_r + \beta_r$.

By the definitions of K_j and k, we have

$$k(t) N = K_i(t) + K_r(t).$$

The assumption that the labor force is always fully employed is given by

$$N_i(t) + N_r(t) = N.$$

We thus built the model.

4.5.2 The motion of the monetary growth model with research

From $m = \chi c$ and equations (4.5.6), we have

$$m = \chi c = \frac{\chi \xi \bar{y}}{1 + \chi \pi}. \tag{4.5.11}$$

Inserting $\pi = \mu - \dot{m}/m$ in equation (4.5.11), we have

$$\dot{m} = \mu_0 m - (\xi + r)k - \xi w, \tag{4.5.12}$$

where we use the definition of $\bar{y}$ and

$$\mu_0 \equiv \frac{1}{\chi} + \mu - \xi.$$

We require $\mu_0 > 0$. As r and w are functions of k_i and Z, $\dot{m}$ is a function of m, k, k_i, and Z.

From $w = \beta \bar{\tau} Z^{\vartheta/\beta} f$ in equations (4.5.2) and $w = \beta_r G/\alpha_0 N_r$ in equations (4.5.10), we solve

$$\alpha_0 N_r \beta \bar{\tau} Z^{\vartheta/\beta} f = \beta_r \tau Z^{\vartheta/\beta} N_i f + \beta_r \mu N m,$$

where we use $G = \tau Z^{\vartheta/\beta} N_i f + \mu N m$. From this equation and $N_i + N_r = N$, we solve the labor distribution as functions of m, k_i, and Z as follows

$$N_i = n_i - \frac{n_0 m}{Z^{\vartheta/\beta} f}, \quad N_r = n_r + \frac{n_0 m}{Z^{\vartheta/\beta} f}, \tag{4.5.13}$$

where

$$n_i \equiv \frac{\alpha_0 N \beta \bar{\tau}}{\beta_r \tau + \alpha_0 \beta \bar{\tau}} > 0, \quad n_r \equiv \frac{\beta_r \tau N}{\beta_r \tau + \alpha_0 \beta \bar{\tau}} > 0, \quad n_0 \equiv \frac{\beta_r \mu N}{\beta_r \tau + \alpha_0 \beta \bar{\tau}} > 0.$$

Substituting $G = \tau Z^{\vartheta/\beta} N_i f + \mu N m$ and $r + \delta_k = \alpha \bar{\tau} f / k_i$ into the equation for K_r in (4.5.10), we obtain

$$K_r = \frac{n_i \tau \alpha_r}{\alpha_0 \alpha \bar{\tau}} k_i Z^{\vartheta/\beta} - \frac{(\tau n_0 - \mu N)\alpha_r}{\alpha_0 \alpha \bar{\tau} A} k_i^{\beta} m. \tag{4.5.14}$$

From the definition of k_i, we have

$$K_i = n_i k_i Z^{\vartheta/\beta} - \frac{n_0 k_i^{\beta} m}{A}.$$

We see that K_i and K_r are functions of m, k_i, and Z. Substituting $K_i = k_i Z^{\vartheta/\beta} N_i$ and equation (4.5.14) into $kN = K_i + K_r$ yields

$$k = \Lambda_0(k_i, Z, m) \equiv \alpha_Z Z^{\vartheta/\beta} k_i - \alpha_m m k_i^{\beta}, \tag{4.5.15}$$

where we also use the equation for N_i in (4.5.13) and

$$\alpha_Z \equiv \frac{(\alpha_0 \alpha \bar{\tau} + \alpha_r \tau) n_i}{\alpha_0 \alpha \bar{\tau} N} > 0, \quad \alpha_m \equiv \frac{\alpha_0 \alpha \bar{\tau} n_0 + \alpha_r \tau n_0 - \alpha_r \mu N}{\alpha_0 \alpha \bar{\tau} NA}.$$

Equation (4.5.15) determines k as a unique function of m, k_i, and Z. It can be seen that all the variables can be now expressed as functions of m, k_i, and Z at any point of time.

Insert equations (4.5.15) and (4.5.1) in equation (4.5.12)

$$\dot{m} = \Lambda_m(k_i, Z, m) \equiv \mu_0 m - \left(\xi - \delta_k + \frac{\alpha \bar{\tau} f}{k_i}\right) \Lambda_0 - \xi \beta \bar{\tau} Z^{\vartheta/\beta} f. \tag{4.5.16}$$

Substituting $F = Z^{\vartheta/\beta} N_i f$ into equation (4.5.8), we obtain

$$\dot{Z} = \Lambda_Z(k_i, Z, m) \equiv \upsilon_i Z^{-\vartheta_i + \vartheta/\beta} N_i f + \upsilon_r Z^{\vartheta_r} K_r^{\alpha_r} N_r^{\beta_r} - \delta_z Z. \tag{4.5.17}$$

Substituting $a = k + m$ and $s = \lambda \bar{y}$ into equation (4.5.7), we obtain

$$\dot{k} = \left(\frac{\alpha \lambda \bar{\tau} f}{k_i} - \lambda \delta_k - \xi\right) k + \lambda \beta \bar{\tau} Z^{\vartheta/\beta} f - \xi m - \Lambda_m, \tag{4.5.18}$$

where we use equation (4.5.16) and

$$\bar{y} = \left(1 - \delta_k + \frac{\alpha \bar{\tau} f}{k_i}\right) k + \beta \bar{\tau} Z^{\vartheta/\beta} f + m. \tag{4.5.19}$$

Taking derivatives of equation (4.5.15) with respect to t, we obtain

$$\dot{k} = \frac{\alpha_Z \vartheta}{\beta} Z^{\vartheta/\beta - 1} k_i \Lambda_Z - \alpha_k k_i^{\beta} \Lambda_m + \left(\alpha_Z Z^{\vartheta/\beta} - \alpha_m \beta m k_i^{-\alpha}\right) \dot{k}_i.$$

Substituting this equation and equation (4.5.15) into equation (4.5.18) yields

$$\begin{aligned} \dot{k}_i &= \Lambda_{k_i}(k_i, Z, m) \\ &\equiv \frac{\Lambda_1 \Lambda_0 + \lambda \beta \bar{\tau} Z^{\vartheta/\beta} f - \xi m + \left(\alpha_m k_i^{\beta} - 1\right) \Lambda_m - \alpha_Z \vartheta Z^{\vartheta/\beta - 1} k_i \Lambda_Z / \beta}{\alpha_Z Z^{\vartheta/\beta} - \alpha_m \beta m k_i^{-\alpha}}, \end{aligned} \tag{4.5.20}$$

where $\Lambda_1 \equiv \alpha \lambda \bar{\tau} f / k_i - \lambda \delta_k - \xi$.

Summarizing the analytical results in this section, we obtain the following lemma.

Lemma 4.5.1
The motion of $m(t)$, $Z(t)$, and $k_i(t)$ is determined by the three differential equations

$$\dot{m} = \Lambda_m(k_i, Z, m),$$
$$\dot{Z} = \Lambda_H(k_i, Z, m),$$
$$\dot{k}_i = \Lambda_{k_i}(k_i, Z, m). \tag{4.5.21}$$

For any positive $m(t), Z(t)$, and $k_i(t)$, all the other variables are uniquely determined at any point of time by the following procedure: k by (4.5.15) $\rightarrow$ $K = kN \rightarrow N_i$ and N_v by (4.5.13) $\rightarrow K_i = k_i Z^{\vartheta/\beta} N_i \rightarrow K_r$ by (4.5.14) $\rightarrow$ r and w by (4.5.2) $\rightarrow \bar{y}$ by (4.5.5) $\rightarrow \pi = \mu - \Lambda_m/m \rightarrow c$ and s by (4.5.6) $\rightarrow$ $a = k + m \rightarrow F = AZ^{\vartheta} K_i^{\alpha} N_i^{\beta}$.

The lemma is important in that it provides a procedure for computing the motion of the entire system. For any proper initial conditions, it is straightforward to simulate the model with computer. As the expressions are tedious, we do not discuss under what conditions the dynamic system has meaningful solutions. We now study steady states.

At steady state we have $\pi = \mu$ and $a = \lambda\bar{y}$. From $a = \lambda\bar{y}$ and equation (4.5.19), we have

$$k = \frac{\lambda\beta\bar{\tau} Z^{\vartheta/\beta} f - \xi m}{\xi + \delta_k \lambda - \lambda\alpha\bar{\tau} A k_i^{-\beta}}. \tag{4.5.22}$$

From $m = \chi c, c = \xi\bar{y}/(1 + \chi\mu)$, and $a = \lambda\bar{y}$, we have $k = \mu_0 m$, where

$$\mu_0 \equiv \left(\frac{1}{\chi} + \mu\right)\frac{\lambda}{\xi} - 1.$$

We require $\mu_0 > 0$. From $k = \mu_0 m$ and equation (4.5.21), we solve

$$m = \frac{\lambda\beta\bar{\tau} Z^{\vartheta/\beta} f}{\left(\xi + \delta_k\lambda - \lambda\alpha\bar{\tau} A k_i^{-\beta}\right)\mu_0 + \xi}. \tag{4.5.23}$$

From $k = \mu_0 m$ and (4.5.15), we solve

$$m = \frac{\alpha_Z Z^{\vartheta/\beta} k_i}{\mu_0 + \alpha_m k_i^{\beta}}. \tag{4.5.24}$$

From equations (4.5.21) and (4.5.22) we solve

$$k_i = \left[\frac{(\beta + \alpha\alpha_Z)\lambda\mu_0\bar{\tau} A}{\xi\mu_0\alpha_Z + \delta_k\lambda\mu_0\alpha_Z + \xi\alpha_Z - \alpha_m\lambda\beta\bar{\tau} A}\right]^{1/\beta}. \tag{4.5.25}$$

From the definitions of the parameters, the numerator, $(\beta+\alpha\alpha_Z)\lambda\mu_0\overline{\tau}A$, is positive. The sign of the denominator is given by

$$\xi\mu_0\alpha_Z+\delta_k\lambda\mu_0\alpha_Z+\xi\alpha_Z-\alpha_m\lambda\beta\overline{\tau}A$$
$$=\left[(\xi\mu_0+\delta_k\lambda\mu_0+\xi)\alpha_0-\lambda\beta_r\mu\right]\frac{((\alpha_0\alpha\overline{\tau}+\alpha_r\tau))\beta}{(\beta_r\tau+\alpha_0\beta\overline{\tau})\alpha\alpha_0}+\frac{\alpha_r\mu\lambda\beta}{\alpha_0\alpha}.$$

It is not strict to require this term to be positive. For instance, if $(\alpha_r+\beta_r)\xi\geq\lambda\beta_r\mu$,[30] then the requirement is satisfied.

Substituting equation (4.5.22) into the labor and capital distribution equations, we have

$$N_i=n_i-\frac{n_0\alpha_Z k_i}{\left(\mu_0+\alpha_m k_i^{\beta}\right)f},\quad N_r=n_r+\frac{n_0\alpha_Z k_i}{\left(\mu_0+\alpha_m k_i^{\beta}\right)f},$$
$$K_r=A_rZ^{\vartheta/\beta},\quad K_i=A_iZ^{\vartheta/\beta},\tag{4.5.26}$$

where

$$A_r\equiv\left[\frac{n_i\tau}{\alpha_0\alpha\overline{\tau}}-\frac{(\tau n_0-\mu N)\alpha_Z k_i^{\beta}}{\left(\mu_0+\alpha_m k_i^{\beta}\right)\alpha_0\alpha\overline{\tau}A}\right]\alpha_r k_i,$$
$$A_i\equiv\left(n_i-\frac{\alpha_Z n_0 k_i^{\beta}}{\left(\mu_0+\alpha_m k_i^{\beta}\right)A}\right)k_i.$$

As k_i is already determined, N_j and A_j are determined. From equations (4.5.17) and (4.5.26), we get

$$\Omega(Z)=\upsilon_i N_i f Z^{x_i}+\upsilon_r A_r^{\alpha_r}N_r^{\beta_r}Z^{x_r}-\delta_z=0,\tag{4.5.27}$$

where

$$x_i\equiv\frac{\vartheta}{\beta}-\vartheta_i-1,\quad x_r\equiv\vartheta_r+\frac{\alpha_r\vartheta}{\beta}-1.$$

This equation is mathematically identical to equation (4.5.27) in Section 3.7. As Lemma 3.7.1, we have the following lemma.

Lemma 4.5.2

(1) If $x_r<0$ and $x_i<0$, the system has a unique stable equilibrium; (2) If $x_r>0$ and $x_i>0$, the system has a unique equilibrium; and (3) If $x_r>0$ and $x_i<0$ ($x_r<0$ and $x_i>0$), the system has either no equilibrium, one equilibrium, or two equilibrium points. For a solution Z, all the other variables are given by following procedure: k_i by (4.5.25) $\rightarrow$ m by (4.5.22) $\rightarrow$ $k=\mu_0 m\rightarrow K=kN\rightarrow N_j$,

$j = i, r$, by (4.5.13) $\rightarrow K_r$ by (4.5.14) $\rightarrow K_i = k_i Z^{\vartheta/\beta} N_i \rightarrow r$ and w by (4.5.2) $\rightarrow$ $\pi = \mu - \Lambda_m/m \rightarrow a = k + m \rightarrow F = AZ^{\vartheta} K_i^{\alpha} N_i^{\beta} \rightarrow \bar{y}$ by (4.5.5) $\rightarrow c$ and s by (4.5.6).

Lemma 3.7.1 provides a procedure to determine the equilibrium values of all the variables.

4.5.3 The inflation rate and the economic structure

This section examines effects of change in μ. Taking derivatives of equation (4.5.25) with respect to μ yields

$$\frac{dk_i}{d\mu} = \frac{\lambda}{\beta\xi\mu_0 k_i} + \frac{\lambda}{(\beta + \alpha\alpha_Z) f \beta\lambda\mu_0 \bar{\tau}} \left[\frac{\alpha_m \beta \bar{\tau} A}{\mu} - (\xi + \delta_k \lambda) \frac{\alpha_Z}{\xi} \right].$$

A tedious calculation with substituting the value of k_i and the parameters into the above equation shows

$$\frac{dk_i}{d\mu} = \frac{\alpha_r \lambda \beta k_i^{\beta} \left[\xi + (\beta_r \alpha/\alpha_r - \beta) \bar{\tau} \lambda/\chi \right]}{(\beta + \alpha\alpha_Z)(\beta_r \tau + \alpha_0 \beta \bar{\tau}) \alpha\lambda\xi \bar{\tau} \mu_0^2 A}.$$

We see that the direction of the impact on k_i is the same as that of $\xi + (\beta_r \alpha/\alpha_r - \beta) \bar{\tau} \lambda/\chi$. Hence, if $\beta_r/\alpha_r \geq \beta/\alpha$, then k_i definitely rises. If $\beta_r/\alpha_r < \beta/\alpha$ but $\beta_r \alpha/\alpha_r - \beta$ is small (and, for instance, with χ being near unit and the propensity to save is not very high, then k_i still rises.

From equations (4.5.26), we obtain the effects on the labor distribution as follows

$$\frac{dN_i}{d\mu} = -\frac{dN_r}{d\mu} = -\frac{n_0 \alpha_Z}{\left(\mu_0 k_i^{-\beta} + \alpha_m\right)^2 \mu f} \left(\frac{\beta\mu\mu_0}{k_i} \frac{dk_i}{d\mu} + \frac{\lambda}{\chi\xi} - 1 \right).$$

If $\lambda > \chi\xi$ and $dk_i/d\mu > 0$, then an increase in the inflation shifts some of the labor force from the industrial sector to the university. From equation (4.5.27), we have the impact on Z as follows

$$-\left(\frac{x_i \upsilon_i N_i f Z^{x_i} + x_r \upsilon_r A_r^{\alpha_r} N_r^{\beta_r} Z^{x_r}}{Z} \right) \frac{dZ}{d\mu} = \left(\upsilon_i f Z^{x_i} - \upsilon_r \beta_r A_r^{\alpha_r} N_r^{\beta_r - 1} Z^{x_r} \right) \frac{dN_i}{d\mu}$$

$$- \frac{\alpha \upsilon_i N_i f Z^{x_i}}{k_i} \frac{dk_i}{d\mu} + \upsilon_r \beta_r A_r^{\alpha_r - 1} N_r^{\beta_r} Z^{x_r} \frac{dA_r}{d\mu},$$

where

$$\frac{dA_r}{d\mu} = \left[\frac{A_r}{k_i} - \frac{(\tau n_0 - \mu N) \alpha_Z \alpha_r \beta k_i^{\beta}}{\left(\mu_0 + \alpha_m k_i^{\beta}\right) \alpha_0 \alpha \bar{\tau} A} \right] \frac{dk_i}{d\mu} - \frac{(\tau n_0/\mu - N) \alpha_Z \alpha_r k_i^{1+\beta}}{\left(\mu_0 + \alpha_m k_i^{\beta}\right) \alpha_0 \alpha \bar{\tau} A}$$

$$+ \frac{(\tau n_0 - \mu N) \alpha_r k_i^{1+\beta}}{\left(\mu_0 + \alpha_m k_i^{\beta}\right)^2 \alpha_0 \alpha \bar{\tau} A} \left(\frac{\lambda}{\xi} + \frac{\alpha_m k_i^{\beta}}{\mu} + \alpha_m \beta k_i^{-\alpha} \frac{dk_i}{d\mu} \right).$$

The sign of the right-hand side of the equation is ambiguous. The term, $x_i \upsilon_i N_i f Z^{x_i} + x_r \upsilon_r A_r^{\alpha_r} N_r^{\beta_r} Z^{x_r}$, in the left-hand side is either positive or negative, depending on the combinations of the signs of x_i and x_r. It is straightforward to confirm: (1) if $x_r < 0$ and $x_i < 0$, the term is negative; (2) if $x_r > 0$ and $x_i > 0$, it is positive; and (3) if $x_r > 0$ and $x_i < 0$ ($x_r < 0$ and $x_i > 0$) and the system has two steady state points, it is positive at the steady state with the higher level of Z and negative at the other steady state.

As the impact on knowledge is difficult to explicitly determine, it is difficult to explicitly determine the signs of the effects on the other variables. From equations (4.5.2), we have

$$\frac{dr}{d\mu} = -\alpha\beta\bar{\tau}Ak_i^{-\beta-1}\frac{dk_i}{d\mu}, \quad \frac{dw}{d\mu} = \frac{\alpha w}{k_i}\frac{dk_i}{d\mu} + \vartheta w\bar{\tau}Z^{\vartheta/\beta-1}f\frac{dZ}{d\mu}.$$

If k_i and Z are increased, then the rate of interest falls and the wage rate rises. In general, it is difficult to judge the impact on the wage rate.[31]

As we have explicitly provided the procedure to determine all the variables, it is straightforward to give the expressions of the effects of change in any parameter upon all the variables. As the expressions are too complicated, we will not analyze the model any further. It should be remarked that like in Section 3.7, we might also simulate the model as the procedure for computing the motion of all the variables are also explicitly given.

4.6 Money, public goods, and returns to scale

Infrastructure is public investment in roads, sewers, airports, and even includes education. Infrastructures provide widespread benefits to consumers and raises the return on private investment and productivity of production sectors. It is argued that recent inadequate public investment in infrastructure has been the major cause of the productivity slowdown in the United States. In 1991, Barro found a negative and significant effect of the level of public consumption as a percentage on economic growth rates of countries.[32] Although he asked for less government intervention, Adam Smith never said that government should not intervene in economic activity at all. The problem is not only how much the government intervenes but also how it spends its resources. Government can affect both physical capital formation and human capital accumulation. Government's expenditures on health, nutrition, and education can be considered as an investment in the sense that they will lead to a more productive force in the future. The quantity and quality of public infrastructure – roads, airports, railways, ports, military defense systems, schools, public hospitals, and the like – are essential for the welfare of people and efficiency of production. For an economy as a whole, formation of government capital is not necessarily conductive to economic growth as it abstracts private use of capital. It is important to show under what conditions an increase in formation of public infrastructures would stimulate economic growth.

4.6.1 A monetary growth model with public goods

This section will introduce (public) infrastructure into the OSG model.[33] Production function now includes, together with private inputs, those government inputs that will affect the level of output. For simplicity, we only consider a single type of public physical capital. The economy consists of industrial and public sectors. The industrial sector is the same as in the OSG model, except that infrastructures will affect productivity. The population is constant and homogenous; each worker is employed in either of the two sectors. The production function with public goods is

$$F(t) = AG^{v}(t)K_i^{\alpha_i}(t)N_i^{\beta_i}(t), \alpha_i + \beta_i = 1, \quad A,\ \alpha_i,\ \beta_i,\ v > 0,$$

where A, α_i, β_i, and v are parameters, and

$F(t)$: output of the industrial sector at time t;
$K_i(t)$: capital stocks employed by the industrial sector;
$N_i(t)$: labor force employed by the industrial sector; and
$G(t)$: (service) level of the public sector.

We interpret the variable, $G(t)$, as public goods such as physical and institutional infrastructures. The aggregate public goods is supplied by the government and is taken as given by the firms. Despite increasing social returns to scale, the function allows to maintain the assumption of perfect competition in the goods market since the technology exhibits constant returns to scale for any given level of public goods, which firms cannot control. It is reasonable to consider that production efficiency will be improved as the service level of the public sector is improved, it seems that doubling the service level will not double the output with fixed $K_i(t)$ and $N_i(t)$. That is, the parameter, v, should be less than one. Because public service is freely available to firms, it is not a decision variable for the industrial sector.

The government finances the public sector by imposing taxes on, for instance, the outputs of the firms, the capital income, the labor income, and consumptions. We assume that the government imposes taxes only on the firms.[34] Introduce

$$k_i(t) \equiv \frac{K_i(t)}{N_i(t)}, \quad f(t) \equiv \frac{F(t)}{N_i(t)}.$$

The marginal conditions are given by

$$r(t) + \delta_k = \frac{\overline{\tau}\alpha_i f(t)}{k_i(t)}, \quad w(t) = \overline{\tau}\beta_i f(t), \tag{4.6.1}$$

where $r(t)$ is the rate of interest, $w(t)$ is the wage rate, and $\overline{\tau} \equiv 1 - \tau$ (where τ is a fixed tax rate on the producers, $1 > \tau \geq 0$).

As in Section 4.1, we have

$$\dot{M}(t) = \mu M(t), \quad \frac{\dot{M}(t)}{P(t)} = \mu m(t), \quad M(t) \geq \chi P(t) c(t), \tag{4.6.2}$$

where χ is a positive parameter. We require $0 < \chi < 1$. As in Section 4.1, the budget constraint is given by

$$(1 + \chi\pi(t))c(t) + s(t) = \bar{y}(t) = r(t)k(t) + w(t) + a(t), \tag{4.6.3}$$

where $a \equiv k + m$ and $\pi = \mu - \dot{m}/m$, and we use $m(t) = \chi c(t)$. With $U = c^{\xi_0} s^{\lambda_0}$, the optimal solution is

$$c(t) = \frac{\xi \bar{y}(t)}{1 + \chi\pi(t)}, \quad s(t) = \lambda \bar{y}(t), \tag{4.6.4}$$

where

$$\xi \equiv \frac{\xi_0}{\xi_0 + \lambda_0}, \quad \lambda \equiv \frac{\lambda_0}{\xi_0 + \lambda_0}.$$

The wealth accumulation is given by

$$\dot{a}(t) = s(t) - a(t). \tag{4.6.5}$$

We now describe the public sector. In this model, we assume that the public sector is financially supported by the government. The capital stocks and workers employed by the public sector are paid at the same rates that the private sector pays the services of these factors. The government may have various objectives in providing public services. It is commonly assumed that the government will maximize the social welfare over a certain period of time in determining scales and scopes of its intervention. The public sector is assumed to behave effectively in the sense that it will use the income to maximize public services. We neglect possible inefficiency, such as official corruption, of the public sector. The production function of the public sector is given by

$$G(t) = K_p^{\alpha_{p0}}(t) N_p^{\beta_{p0}}(t), \quad \alpha_{p0}, \beta_{p0} > 0,$$

where $K_p(t)$ is the capital stocks and $N_p(t)$ is the labor force employed by the public sector. The public sector is faced with the budget constraint

$$w(t)N_p(t) + (r(t) + \delta_k)K_p(t) = Y_g(t) \equiv \tau F(t) + \mu m(t)N.$$

The optimal solution is given by

$$(r + \delta_k)K_p = \alpha_p Y_g, \quad wN_p = \beta_p Y_g, \tag{4.6.6}$$

where

$$\alpha_p \equiv \frac{\alpha_{p0}}{\alpha_{p0}+\beta_{p0}}, \quad \beta_p \equiv \frac{\beta_{p0}}{\alpha_{p0}+\beta_{p0}}.$$

The factors are fully employed

$$N_i(t)+N_p(t)=N, \quad K_i(t)+K_p(t)=K(t).$$

We have thus built the model. Before analyzing the behavior of the monetary economy, we summarize the main analytical results by Zhang in the following theorem.

Theorem 4.6.1[35]
Assume $\chi = 0$. The dynamics of the economic system are governed by the one-dimensional differential equation. The dynamic system has a unique equilibrium. Moreover, if $x < (>)0$, where $x \equiv \alpha_i + \alpha_p v - 1$, the unique equilibrium is stable (unstable).

We illustrate the stability conditions by Figure 4.6.1.[36]

The parameter, x, determines stability of the system. From $x = \alpha_i + \alpha_p v - 1$, we see that this parameter measures net increasing returns to scale in capital accumulation. For instance, let us examine what happens to the capital accumulation equation when we double capital stocks. If we double the capital stocks, then capital inputs in the two sectors are exactly doubled because of the Cobb-Douglas technologies and perfectly competitive input factor markets. The level of public service is 2^{α_p} times as many as the old level of services because capital input is doubled. This means that productivity of the industrial sector is $2^{\alpha_p v}$ times as

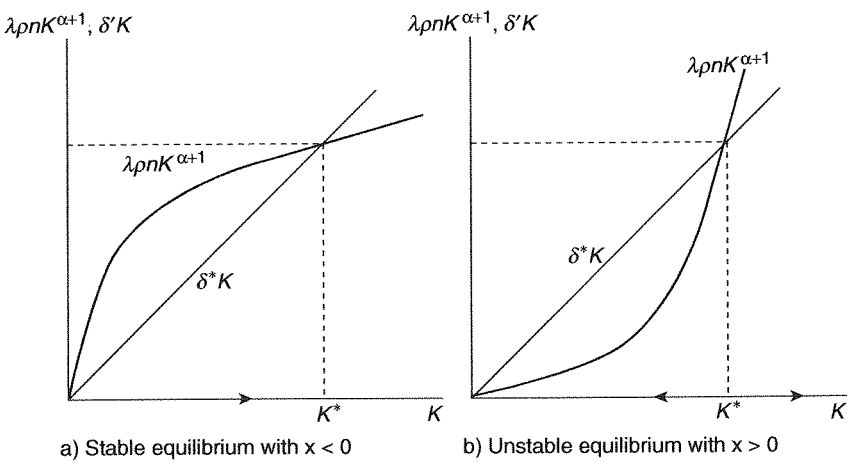


Figure 4.6.1 Stable and unstable steady states.

high as the old level. On the other hand, the output will increase 2^{α_i} times as many as the initial output because of an increase in capital stocks in the industrial sector. Consequently, as the total capital doubles, the total output will be $2^{\alpha_i+\alpha_p v}$ times as many as before. We see that if $\alpha_i + \alpha_p v < (>)1$ (that is, $x < (>)0$), the capital accumulation exhibits increasing (decreasing) returns in scale. If the capital accumulation exhibits decreasing returns to scale, the system becomes stable.

It can be shown that in the case of $x < 0$, the traditional view on savings is valid – an increase in the propensity to save, λ, tends to increase income; in the case of $x > 0$, the steady state is lowed down. The following corollary summarizes the results.

4.6.2 The impact of the inflation policy

From $m = \chi c$ and equations (4.6.4), we have

$$m = \chi c = \frac{\chi \xi \bar{y}}{1 + \chi \pi}. \tag{4.6.7}$$

Inserting $\pi = \mu - \dot{m}/m$ in equation (4.6.7), we have

$$\dot{m} = \mu_0 m - (\xi + r)k - \xi w, \tag{4.6.8}$$

where we use the definition of $\bar{y}$ and $\mu_0 \equiv 1/\chi + \mu - \xi$, (which is assumed to be positive).

From $w = \bar{\tau}\beta_i f$ in equations (4.6.1) and $w = \beta_p Y_g/N_p$ in equations (4.6.6), we solve

$$\frac{\bar{\tau}\beta_i}{\beta_p} f N_p = \tau N_i f + \mu m N,$$

where we use $Y_g = \tau N_i f + \mu m N$. From this equation and $N_i + N_p = N$, we solve the labor distribution as follows

$$N_i = \frac{(\bar{\tau}\beta_i/\beta_p)Nf - \mu m N}{\bar{\tau}_0 f}, \quad N_p = \frac{\tau N f + \mu m N}{\bar{\tau}_0 f}, \tag{4.6.9}$$

where $\bar{\tau}_0 \equiv \bar{\tau}\beta_i/\beta_p + \tau$. Substituting $Y_g = \tau N_i f + \mu m N$ and $r + \delta_k = \alpha_i \bar{\tau} f/k_i$ into the equation for K_r in (4.6.6), we obtain

$$K_p = (\tau f + \mu m)\frac{\alpha_p k_i N}{\alpha_i \tau_0 f}, \tag{4.6.10}$$

where $\tau_0 \equiv \bar{\tau} + \tau\beta_p/\beta_i = \beta_p \bar{\tau}_0/\beta_i$. From the definition of k_i, we have

$$K_i = \left(\frac{\bar{\tau}\beta_i Nf}{\beta_p} - \mu m N\right)\frac{k_i}{\bar{\tau}_0 f}. \tag{4.6.11}$$

Substituting N_p in (4.6.9) and K_p in (4.6.10) into $G = K_p^{\alpha_{p0}} N_p^{\beta_{p0}}$, we have

$$\Omega_0(G) \equiv G - \alpha_0 k_i^{\alpha_{p0}} \left(\tau + \frac{\mu m}{A G^v k_i^{\alpha_i}} \right)^{\alpha} = 0, \tag{4.6.12}$$

where we use $f = A G^v k_i^{\alpha_i}$ and

$$\alpha \equiv \alpha_{p0} + \beta_{p0}, \quad \alpha_0 \equiv \left(\frac{\alpha_p N}{\tau_0 \alpha_i} \right)^{\alpha_{p0}} \left(\frac{N}{\overline{\tau}_0} \right)^{\beta_{p0}}.$$

It is straightforward to check that for any positive m and k_i as

$$\Omega_0(0) < 0, \ \Omega_0(\infty) > 0, \ \Omega_0'(G) > 0,$$

equation (4.6.12) has a unique positive solution. We denote this unique relation by $G = \tilde{G}(k_i, m)$. We have

$$\frac{d\tilde{G}}{dk_i} = \alpha_0 k_i^{\alpha_{p0}-1} G_0 \left(\tau + \frac{\mu m}{A G^v k_i^{\alpha_i}} \right)^{\alpha-1} \left[\alpha_{p0} \tau + (\beta_i \alpha_{p0} - \beta_{p0} \alpha_i) \frac{\mu m}{A \tilde{G}^v k_i^{\alpha_i}} \right],$$

$$\frac{d\tilde{G}}{dm} = \frac{\alpha \alpha_0 \mu k_i^{\alpha_{p0}-\alpha_i} G_0}{A \tilde{G}^v} \left(\tau + \frac{\mu m}{A \tilde{G}^v k_i^{\alpha_i}} \right)^{\alpha-1} > 0, \tag{4.6.13}$$

where

$$G_0 \equiv \left[1 + \frac{\mu v m}{A \tilde{G}^{v-1}} \alpha \alpha_0 k_i^{\alpha_{p0}-\alpha_i} \left(\tau + \frac{\mu m}{A \tilde{G}^v k_i^{\alpha_i}} \right)^{\alpha-1} \right]^{-1} > 0.$$

Public service is increased when the real money balance is increased. If $\alpha_{p0}/\beta_{p0} \geq \alpha_i/\beta_i$, as the capital intensity of the production sector rises, the real money balance rise. If $\alpha_{p0}/\beta_{p0} < \alpha_i/\beta_i$, as the capital intensity of the production sector rises, the real money balance may either rise or fall. As G is a function of k_i and m, we see that the per worker output is related to the capital intensity and real money balances as follows

$$f(k_i, m) = A k_i^{\alpha_i} \tilde{G}^v(k_i, m).$$

From the above analyses, we see that r, w, the labor distribution and work force distribution can now be considered as functions of k, k_i, and m. From equations (4.6.1) and (4.6.8), we have

$$\dot{m} = \Lambda_m(k_i, m) \equiv \mu_0 m - \left(\xi - \delta_k + \overline{\tau} \alpha_i A k_i^{-\beta_i} \tilde{G}^v \right) k - \xi \overline{\tau} \beta_i A k_i^{\alpha_i} \tilde{G}^v. \tag{4.6.14}$$

Substituting $K_i = k_i N_i$ and equation (4.6.14) into $kN = K_i + K_p$ yields

$$k = \Lambda_0(k_i, m) \equiv \alpha_k k_i + \alpha_m \frac{k_i^{\beta_i} m}{\tilde{G}^{\nu}}, \tag{4.6.15}$$

where we also use the equation for N_i in (4.6.9) and

$$\alpha_k \equiv \left(\overline{\tau} + \frac{\alpha_p \tau}{\alpha_i}\right)\frac{1}{\tau_0}, \quad \alpha_m \equiv \left(\frac{\alpha_p}{\alpha_i} - \frac{\beta_p}{\beta_i}\right)\frac{\mu}{\tau_0 A}.$$

Equation (4.5.15) determines k as a unique function of k_i and m. It can be seen that all the variables can be now expressed as functions of k_i and m at any point of time. Taking derivatives of equation (4.6.15) with respect to t yields

$$\dot{k} = \overline{\Lambda}_0 \dot{k}_i + \left(1 - \frac{\nu m}{\tilde{G}^{\nu}}\right)\frac{\alpha_m k_i^{\beta_i} \Lambda_m}{\tilde{G}^{\nu}}, \tag{4.6.16}$$

where

$$\overline{\Lambda}_0(k_i, m) \equiv \left(\alpha_k + \frac{\alpha_m \beta_i m}{k_i^{\alpha_i} \tilde{G}^{\nu}} - \alpha_m \nu \frac{k_i^{\beta_i} m}{\tilde{G}^{\nu+1}} \frac{\partial \tilde{G}}{\partial k_i}\right).$$

Substituting $a = k + m$ and $s = \lambda \bar{y}$ into equation (4.6.5), we obtain

$$\dot{k} = \left(\frac{\overline{\tau}\lambda\alpha_i f}{k_i} - \xi - \lambda\delta_k\right)k + \overline{\tau}\lambda\beta_i f - \xi m - \Lambda_m, \tag{4.6.17}$$

where we use equation (4.6.14) and

$$\bar{y} = \left(1 - \delta_k + \frac{\overline{\tau}\alpha_i f}{k_i}\right)k + \overline{\tau}\beta_i f + m. \tag{4.6.18}$$

Substituting this equation and equations (4.5.15) and (4.3.16) into equation (4.5.18) yields

$$\dot{k}_i = \Lambda_{k_i}(k_i, m) \equiv \left[\left(\frac{\overline{\tau}\lambda\alpha_i f}{k_i} - \xi - \lambda\delta_k\right)\Lambda_0 + \overline{\tau}\lambda\beta_i f - \xi m - \Lambda_m - \left(1 - \frac{\nu m}{\tilde{G}^{\nu}}\right)\frac{\alpha_m k_i^{\beta_i}\Lambda_m}{\tilde{G}^{\nu}}\right]\frac{1}{\overline{\Lambda}_0}. \tag{4.6.19}$$

Summarizing the analytical results in this section, we obtain the following lemma.

Lemma 4.6.1
The motion of $m(t)$ and $k_i(t)$ is determined by the two differential equations

$$\dot{m} = \Lambda_m(k_i, m),$$
$$\dot{k}_i = \Lambda_{k_i}(k_i, m). \tag{4.6.20}$$

For any positive $m(t)$ and $k_i(t)$, all the other variables are uniquely determined at any point of time by the following procedure: G by (4.6.12) $\to f = Ak_i^{\alpha_i}\tilde{G}^{\nu} \to k$ by (4.6.15) $\to \bar{y}$ by (4.6.18) $\to K = kN \to K_p$ by (4.6.10) $\to K_i$ by (4.6.11) $\to N_i$ and N_p by (4.6.9) $\to \pi = \mu - \Lambda_m/m \to r$ and w by (4.6.1) $\to\to c$ and s by (4.6.4) $\to a = k + m \to F = fN_i$.

The lemma is important in that it provides a procedure for computing the motion of the entire system. For any proper initial conditions, it is straightforward to simulate the model with computer. As the expressions are tedious, we do not discuss under what conditions the dynamic system has meaningful solutions. We now study steady states.

At steady state we have $\pi = \mu$ and $a = \lambda\bar{y}$. From $a = \lambda\bar{y}$ and equation (4.6.18), we have

$$k = \frac{\lambda\bar{\tau}\beta_i f - \xi m}{\xi + \lambda\delta_k - \lambda\bar{\tau}\alpha_i f/k_i}. \tag{4.6.21}$$

From $m = \chi c$, $c = \xi\bar{y}/(1 + \chi\mu)$, and $a = \lambda\bar{y}$, we have

$$k = \mu_0 m,$$

where $\mu_0 \equiv (1/\chi + \mu)\lambda/\xi - 1$. We require $\mu_0 > 0$. From $k = \mu_0 m$ and equation (4.6.21), we solve

$$\left(\xi_0 - \frac{\mu_0\alpha_i f}{\beta_i k_i}\right) m = f, \tag{4.6.22}$$

where

$$\xi_0 \equiv \frac{(\xi + \lambda\delta_k)\mu_0 + \xi}{\lambda\bar{\tau}\beta_i}.$$

For simplicity we specify $\alpha_{p0} + \beta_{p0} = 1$ and $\nu = 1$. From equation (4.6.12) we solve

$$m = \left(\frac{G}{\alpha_0 k_i^{\alpha_{p0}}} - \tau\right)\frac{AGk_i^{\alpha_i}}{\mu}. \tag{4.6.23}$$

Substituting this equation into equation (4.6.22), we have

$$G^2 - f_1 G + f_2 = 0,$$

where

$$f_1 \equiv \frac{\beta_i \xi_0 k_i^{\beta_i} + \alpha_0 \tau \mu_0 \alpha_i A k_i^{\alpha_{p0}}}{\mu_0 \alpha_i A} > 0, \quad f_2 \equiv \frac{(\tau \xi_0 + \mu)\alpha_0 \beta_i k_i^{\alpha_{p0}+\beta_i}}{\mu_0 \alpha_i A} > 0.$$

The problem has two positive solutions

$$G = \frac{f_1 \pm \sqrt{f_1^2 - 4f_2}}{2}. \tag{4.6.24}$$

We should require $f_1^2 > 4f_1$ for the problem to have at least one meaningful solution. It can be seen that this condition is satisfied if

$$\left(\beta_i \xi_0 k_i^{\beta_i} - \alpha_0 \tau \mu_0 \alpha_i A k_i^{\alpha_{p0}}\right)^2 > 4\mu_0 \mu \alpha_i \alpha_0 \beta_i A k_i^{\alpha_{p0}+\beta_i}. \tag{4.6.25}$$

As this condition is difficult to interpret, we will not further examine it. Substituting $k = \mu_0 m$ into equation (4.6.15) yields

$$\left(\mu_0 - \frac{\alpha_m k_i^{\beta_i}}{G}\right) m = \alpha_k k_i.$$

Insert equation (4.6.23) into the above equation

$$\left(G - \frac{\alpha_m k_i^{\beta_i}}{\mu_0}\right)\left(G - \tau \alpha_0 k_i^{\alpha_{p0}}\right) = \frac{\alpha_k \mu \alpha_0}{A \mu_0} k_i^{\beta_i + \alpha_{p0}}.$$

As G is a function of k_i, the above equation contains a single variable. To solve it, we first substitute $G^2 = f_1 G - f_2$ into the above equation

$$G = \tilde{\alpha} k_i^{\alpha_{p0}},$$

where we use the definitions of α_k, α_m, and τ_0 and

$$\tilde{\alpha} = \left[\frac{(\beta_i \tau \xi_0 + \mu)}{\alpha_i}\right]\left[\frac{(\xi + \lambda \delta_k)\mu_0 + \xi}{\alpha_i \lambda \overline{\tau}} - \left(\frac{\alpha_p}{\alpha_i} - \frac{\beta_p}{\beta_i}\right)\frac{\mu}{\tau_0}\right]^{-1} \alpha_0.$$

Substituting equation (4.6.25) into $G = \tilde{\alpha} k_i^{\alpha_{p0}}$, we solve

$$\tilde{\alpha} k_i^{\alpha_{p0}} f_1 - 2f_2 - 2\tilde{\alpha}^2 k_i^{2\alpha_{p0}} = 0.$$

Substituting the definitions of f_j into the above equation yields

$$k_i = \left[\left(\frac{2\tilde{\alpha} - \alpha_0 \tau}{\xi_0 \tilde{\alpha} - 2\alpha_0(\tau \xi_0 + \mu)}\right)\frac{A \mu_0 \alpha_i \tilde{\alpha}}{\beta_i}\right]^{1/(\beta_i - \alpha_{p0})}, \tag{4.6.26}$$

where we assume $\beta_i \neq \alpha_{p0}$. We thus solved the values of all the variables at steady state. As the expressions are tedious and it is quite difficult to get explicit interpretations, we will not further examine the model.

4.7 A shopping cost model in the Ramsey approach

This section introduces the shopping cost model which emphasizes the exact specification of the transactions demand for money.[37] In this model, money facilitates the transactions that are necessary for delivery and/or consumption of output. Let $\varphi(m(t))$ stand for a function of real money holdings which is the fraction of real resources that are necessary for the task of facilitating transactions. Real money holdings produce "shopping services" in the sense that the more money is held the more real resources are freed from the transaction task. This means, $\varphi'(m) < 0$. There are decreasing returns to the services that can be provided by holding money. It is further required $\varphi''(m) > 0$.

First, we are concerned by the case that only consumers find the services of money useful. Most variables of the model are the same as the corresponding variables in the Sidrauski model in Section 4.1. The representative household maximizes the following utility function

$$U = \int_0^\infty u(c(t))e^{-\rho t}dt, \tag{4.7.1}$$

subject to the portfolio decision constraint

$$a(t) = k(k) + m(t), \tag{4.7.2}$$

and asset accumulation equation

$$\dot{a}(t) = f(k(t)) + x(t) - (1 + \varphi(m(t)))c(t) - na(t) - \pi(t)m(t). \tag{4.7.3}$$

Here, real money holdings does not affect the utility function. The difference in the accumulation equation in this model and the Sidrauski model is the transactions cost term, $\varphi(m)c$. It is straightforward to confirm that as in the Sidrauski model in Section 4.1, Sidrauski's neutrality result holds for this model.

We now consider an alternative approach to the above model by Dornbusch and Frenkel (1973). We assume that money facilitates transactions necessary to produce output. We consider pecuniary costs associated with individual transactions through a shopping-time technology analogous to that in Saving (1971). Instead of equation (4.7.3), the asset accumulation equation now becomes[38]

$$\dot{a}(t) = (1 - \varphi(m(t)))f(k(t)) + x(t) - c(t) - na(t) - \pi(t)m(t). \tag{4.7.4}$$

The consumer problem is to maximize U subject to (4.7.2) and (4.7.4). In steady state, we have

$$\rho = f'(1 - \varphi),$$
$$f'(1 - \varphi) = -f\varphi' - \pi. \tag{4.7.5}$$

From equations (4.7.5) and $\pi = \mu - n$, we have

$$f''(1-\varphi)\frac{dk}{d\mu} - f'\varphi'\frac{dm}{d\mu} = 0,$$

$$0 = -f\varphi''\frac{dm}{d\mu} - f'\varphi'\frac{dk}{d\mu} - 1.$$

We solve

$$\frac{dk}{d\mu} = -\frac{f'\varphi'}{\varphi''f''f(1-\varphi) + (f'\varphi')^2}.$$

As $f'\varphi'$ is negative, the sign of change in μ is the same as that of $\varphi''f''f(1-\varphi) + (f'\varphi')^2$. If there are no decreasing returns to saving shopping costs by holding money ($\varphi'' = 0$), then higher inflation results in higher steady state capita. In general, the effect is ambiguous.

4.8 Concluding remarks

Another well-known approach to introducing money into growth theory is the so-called time-costs of transactions approach. This approach is based on an idea of a "shopping-time" technology, where there is a margin along which agents can substitute time for money in response to inflation.[39] This margin suggests a direct channel through which monetary policies can affect time distribution among leisure, work time, and transactions. As time distribution is shifted, labor supply will be affected and hence the national economic growth.[40] Although this book will not explore implications of this approach for economic growth, we mention how to introduce the technology into the OSG approach. The completion of the model and the analytical work are left to the reader as an exercise.

Almost all the aspects of the model are the same as in Section 3.5. Denote $T(t), T_h(t)$, and $T_z(t)$ respectively stand for work time, leisure time, and time spent on transactions. The time constraint is expressed by

$$T(t) + T_h(t) + T_z(t) = T_0.$$

Following Love and Wen (1999), we assume that time spent transacting is described by the following technology

$$T_z(t) = \theta_0\left(\frac{c(t)}{m(t)}\right)^{\theta}, \quad \theta_0 > 0,\ \theta > 1,$$

where θ_0 and θ are parameters.[41] Time spent transacting is increasing in consumption expenditure and decreasing in real money balances. $\theta > 1$ implies that

using money for transaction has diminishing returns. If θ_0 equals zero, then money disappears in the model.[42] The budget is represented by

$$\hat{y} = r\bar{k} + wT_0 - wT_h - \theta_0 w\left(\frac{c}{m}\right)^{\theta} - \pi m + \tau + a.$$

Maximizing

$$U(t) = T_h^{\sigma_0}(t)c^{\xi_0}(t)s^{\lambda_0}(t), \quad \sigma_0\xi_0, \lambda_0 > 0,$$

Subject to $c + s = \hat{y}$ yields the demand functions. The rest of the model is almost the same as the model in Section 3.5.

APPENDIX

A.4.1 A growth economy with two types of households

The production aspects of the economic system under consideration are similar to the OSG model in Section 3.3. Let there be two groups of consumers, indexed by j, in the economy. Each type of consumers has a fixed number of the population, denoted by N_j. The aggregated labor force N is given by $N = z_1N_1 + z_2N_2$, where z_j are the level of human capital of group j, $j = 1, 2$. We assume z_j to be fixed. We now study behavior of producers. Let $K(t)$ denote the capital stock at time t and N the flow of labor services used at t for production. We use the conventional production function to describe a relationship between inputs and output. The rate of interest and wage rate are determined by markets. Hence, for any individual firm r and w_j are given at each point of time. The production sector chooses the two variables, K and N, to maximize its profit. Denote $\delta_k (0 \leq \delta_k < 1)$ the depreciation rate of physical capital. The marginal conditions are given by

$$\begin{aligned} &r(t) + \delta_k = f'(k(t)), \\ &w_j(t) = z_j w(t), \quad w(t) = f(k(t)) - k(t)f'(k(t)). \end{aligned} \tag{A.4.1.1}$$

We also have

$$\begin{aligned} &\dot{M}(t) = \mu M(t), \quad \mu > 0. \\ &g(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t), \\ &\pi(t) \equiv \frac{\dot{P}(t)}{P(t)} = \mu - \frac{\dot{m}(t)}{m(t)}. \end{aligned} \tag{A.4.1.2}$$

Household j's disposable income in real terms, $\hat{y}_j(t)$, is

$$\hat{y}_j(t) = r(t)k_j(t) + w_j(t) + \mu m(t) - \pi(t)m_j(t) + a_j(t), \tag{A.4.1.3}$$

where

$$a_j(t) \equiv k_j(t) + m_j(t).$$

We assume that at each point of time consumers' preferences over consumption and saving can be represented by the following utility function

$$U_j(t) = c^{\xi_0}(t)s^{\lambda_0}(t), \quad \xi_0,\ \lambda_0 > 0,$$

where ε_{0j} is called household j's propensity to hold money, ξ_{0j} propensity to consume, and λ_{0j} propensity to own wealth.

The budget constraint is given by

$$c_j(t) + s_j(t) = \hat{y}_j(t).$$

The liquidity constraint of household j is formed as

$$m_j(t) = \chi_j c_j(t), \tag{A.4.1.4}$$

where χ_j is a positive parameter.

Insert equations (A.4.1.3) and (A.4.1.4) into the budget constraints

$$\big(1 + \chi_j \pi(t)\big) c_j(t) + s_j(t) = \bar{y}_j(t) \equiv r(t)k_j(t) + w_j(t) + \mu m(t) + a_j(t). \tag{A.4.1.5}$$

Consumers' problem is to choose current money, consumption, and savings in such a way that utility levels are maximized. Maximizing $U_j(t)$ subject to the budget constraint (A.4.1.5) yields

$$\big(1 + \chi_j \pi(t)\big) c_j(t) = \xi_j \bar{y}_j(t), \quad s_j(t) = \lambda_j \bar{y}_j(t), \tag{A.4.1.6}$$

where

$$\xi_j \equiv \rho_j \xi_{0j}, \quad \lambda_j \equiv \rho_j \lambda_{0j}, \quad \rho_j \equiv \frac{1}{\xi_{0j} + \lambda_{0j}}.$$

The real wealth changes as follows

$$\dot{a}_j(t) = s_j(t) - a_j(t). \tag{A.4.1.7}$$

The total demand for money is equal to the total supply

$$N_0 m(t) = m_1(t)N_1 + m_2(t)N_2. \tag{A.4.1.8}$$

The assumption of full employment of capital is given by

$$K(t) = k_1(t)N_1 + k_2(t)N_2,$$

that is

$$k(t) = n_1 k_1(t) + n_2 k_2(t), \tag{A.4.1.9}$$

where $n_j \equiv N_j/N$. This equation expresses $k(t)$ as a unique function of $k_1(t)$ and $k_2(t)$. We have thus built the model.

The dynamics and steady state

First, we need to find equations which describe motion of the dynamic system. For simplicity, we omit time. First, from equations (A.4.1.4) and (A.4.1.6), we obtain

$$\left(1+\chi_j\pi\right)m_j=\chi_j\xi_j\bar{y}_j.$$

Substituting $\bar{y}_j = Rk_j + w_j + \mu m + m_j$, where $R \equiv 1 + r$, into the above equations, we have

$$m_j=\frac{\xi_j\left(Rk_j+w_j+\mu m\right)}{\bar{\chi}_j+\pi}, \tag{A.4.1.10}$$

where $\bar{\chi}_j \equiv 1/\chi_j - \xi_j$. Multiplying equation $\left(1+\chi_j\pi\right)m_j=\chi_j\xi_j\bar{y}_j$ by N_j and then adding up the resulted two equations, we obtain

$$mN_0=\frac{\xi_1\left(Rk_1+w_1+\mu m\right)}{\bar{\chi}_1+\pi}+\frac{\xi_2\left(Rk_2+w_2+\mu m\right)}{\bar{\chi}_2+\pi},$$

where we use equation (A.4.1.8). Solving the above equation in

$$\pi^2+(\bar{\chi}-W)\pi+W_0=0, \tag{A.4.1.11}$$

where

$$W_j(k_1,k_2,m)\equiv\left(\frac{Rk_j+w_j}{m}+\mu\right)\frac{\xi_j}{N_0},\quad W\equiv W_1+W_2,\ \bar{\chi}\equiv\bar{\chi}_1+\bar{\chi}_2,$$

$$W_0\equiv\bar{\chi}_1\bar{\chi}_2-\bar{\chi}_2W_1-\bar{\chi}_1W_2.$$

It can be seen that if $\bar{\chi}_1=\bar{\chi}_2$, the inflation rate is uniquely determined

$$\pi=W_1+W_2-\bar{\chi}_1. \tag{A.4.1.12}$$

The general solutions to equation (A.4.1.10) are given as

$$\pi_{1,2}=\Pi(k_1,k_2,m)\equiv\frac{\bar{\chi}-W\pm\sqrt{(\bar{\chi}-W)^2-4W_0}}{2}. \tag{A.4.1.13}$$

It is straightforward to confirm:

$$(\bar{\chi}-W)^2-4W_0=\left[\left(\bar{\chi}_1-\bar{\chi}_2\right)-\left(W_1-W_2\right)\right]^2+2W_1W_2>0.$$

Hence, the equation has always solutions. For a solution, π, to be meaningful, we should also require: $1+\chi_j\pi>0$. We will not examine these conditions as the

expressions are so tedious that it is not easy to explicitly interpret behavioral implications. For any meaningful solution π, from equation (A.4.1.13) and $\pi = \mu - \dot{m}/m$, we obtain

$$\dot{m} = \Omega_m(k_1, k_2, m) \equiv m\mu - m\Pi(k_1, k_2, m). \tag{A.4.1.14}$$

As k is a unique function of k_1 and k_2 from equations (A.4.1.1) we see that r and w_j can also be expressed as unique functions of k_1 and k_2. From the definition of $\bar{y}_j, \bar{y}_j$ are functions of k_1, k_2, and m. From equation (A.4.1.10), we have

$$m_j = \Omega_j(k_1, k_2, m) \equiv \frac{\xi_j(Rk_j + w_j + \mu m)}{\bar{\chi}_j + \Pi(k_1, k_2, m)}. \tag{A.4.1.15}$$

Substituting $s_j = \lambda_j \bar{y}_j$ and $a_j = k_j + m_j$ into equations (A.4.1.7), we have

$$\dot{k}_j = (\lambda_j R - 1)k_j + \lambda_j w_j + \lambda_j \mu m - (1 - \lambda_j)m_j - \dot{m}_j, \tag{A.4.1.16}$$

where $R \equiv 1 + r$. Taking derivatives of equations (A.4.1.15) with respect to t yields

$$\dot{m}_j = \frac{\partial \Omega_j}{\partial k_1}\dot{k}_1 + \frac{\partial \Omega_j}{\partial k_2}\dot{k}_2 + \frac{\partial \Omega_j}{\partial m}\Omega_m. \tag{A.4.1.17}$$

Insert equations (A.4.1.17) into equations (A.4.1.16)

$$\begin{aligned} \dot{k}_1 &= \Omega_{1k} - \Omega_{1m}\dot{k}_2, \\ \dot{k}_2 &= \Omega_{2k} - \Omega_{2m}\dot{k}_1, \end{aligned} \tag{A.4.1.18}$$

where

$$\Omega_{1k} \equiv \left[(\lambda_1 R - 1)k_1 + \lambda_1 w_1 + \lambda_1 \mu m - (1 - \lambda_1)m_1 - \frac{\partial \Omega_1}{\partial m}\Omega_m\right]\left(1 + \frac{\partial \Omega_1}{\partial k_1}\right)^{-1},$$

$$\Omega_{2k} \equiv \left[(\lambda_2 R - 1)k_2 + \lambda_2 w_2 + \lambda_2 \mu m - (1 - \lambda_2)m_2 - \frac{\partial \Omega_2}{\partial m}\Omega_m\right]\left(1 - \frac{\partial \Omega_2}{\partial k_2}\right)^{-1},$$

$$\Omega_{1m} \equiv \frac{\partial \Omega_1}{\partial k_2}\left(1 + \frac{\partial \Omega_1}{\partial k_1}\right)^{-1}, \quad \Omega_{2m}(k_1, k_2, m) \equiv \left(1 - \frac{\partial \Omega_2}{\partial k_2}\right)^{-1}\frac{\partial \Omega_2}{\partial k_1}.$$

Solve equations (A.4.1.18)

$$\begin{aligned} \dot{k}_1 &= \frac{\Omega_{1k} - \Omega_{1m}\Omega_{2k}}{1 - \Omega_{1m}\Omega_{2m}}, \\ \dot{k}_2 &= \frac{\Omega_{2k} - \Omega_{1k}\Omega_{2m}}{1 - \Omega_{1m}\Omega_{2m}}. \end{aligned} \tag{A.4.1.19}$$

Lemma A.4.1.1

The motion of the economic system is described by three differential equations, (A.4.1.14) and (A.4.1.19) with $k_1(t), k_2(t)$, and $m(t)$ as variables. All the other variables are determined as functions of $k_1(t), k_2(t)$, and $m(t)$ by the following procedure: $k(t)$ by (A.4.1.9) $\rightarrow m_j(t), j = 1, 2$, by (A.4.1.15) $\rightarrow a_j(t) = k_j(t) + m_j(t) \rightarrow w_j(t)$ and $r(t)$ by (A.4.1.1) $\rightarrow K(t) = k(t)N \rightarrow f(t) = f(k(t)) \rightarrow F(t) = f(t)N \rightarrow \pi(t)$ by (A.4.1.13) $\rightarrow \bar{y}_j(t)$ by (A.4.1.5) $\rightarrow c_j(t)$ and $s_j(t)$ by (A.4.1.6).

5 Unemployment and money

One of the basic questions in economic theory is why modern economies are incapable of employing, fully and efficiently, their available resources. Unemployment of labor force seems to attract the greatest attention of researchers. Unemployed men cost society dearly. They may diminish their human capital through different kinds of "depreciation." A high rate of unemployment is often associated with social unrest and high crime rates. In many countries, high unemployment is persistent. In 1995, for instance, the proportion of unemployed workers who had been unemployed for more than a year exceeded 30 percent in most European countries. It is one thing to be unemployed for a few days or even a few months; but quite another to be unemployed for a few years. Those who become long-term unemployed end up losing skills and work habits.

The history of economic theory shows that it is not an easy matter to theoretically explain growth with unemployment. Speed of changes in variables varies according to institutions and cultural environment. The Walrasian economics assumes instantaneous adaptation of labor market to its equilibrium. In reality, friction exists everywhere; non-frictional motion is rarely observed. A frictional factor in the labor market is the existence of long-term relationships between firms and workers. Many jobs involve long-term attachments and considerable firm-specific skills on the part of workers. For instance, lifetime employment practice in Japanese companies means that market mechanism will not work smoothly in the Japanese labor market. Workers would stay in their current jobs even when they are offered better work, probably short-run, opportunities. Since Adam Smith, there has been a long-traditional belief in economics that freely competitive market can supply the most efficient co-ordination mechanism in optimal allocation of resources. We followed this traditional belief in the previous chapters, showing how market forces should be able to harmonize a number of individual decisions and lead to optimal allocation of resources. In examining free economies we allowed neither "fixed wages" like in the Keynesian economics, nor "a generous social welfare policy" (which may give incentives for some people to remain in the unemployment state) for unemployed people by the government, nor a mere "survival wage" rate like in the Marxian economics, nor "friction" of structural adaptations in changeable knowledge-oriented economies. It is obvious that if we

allowed any of these possible situations to exist in our theoretical models, we could not guarantee full employment as in the previous chapters.

Phillips (1958) found a negative correlation between inflation and the rate of unemployment in British data. The well-known Phillips curve describes a relationship between inflation and unemployment.[1] The negative relationship was initially believed valid; but it has become clear that the relation is held only under certain conditions during some period. Moreover, the Phillips curve is not based on any micro foundation. This chapter is concerned with economic dynamics with unemployment. We propose some models to analyze the complexity of economic evolution with unemployment. It should be noted that the models in this chapter are not inconsistent with the rest of the book in the sense that if we allow the fixed factors to be determined by market mechanisms, then the models in this chapter converge to the models in the rest of the book, and vice versa.

5.1 A traditional model of inflation and unemployment

We now consider an interaction of inflation and unemployment and introduce a few basic hypotheses in the literature of interactions between inflation and unemployment.[2] Denote the rate of inflation $\pi(t)$ (which is defined as $\dot{P}/P$, P being the price). The expectations-augmented version of the Phillips relation assumes the following relationship among the rate of inflation, the unemployment rate, $N_u(t)$, and the expected rate of inflation, $\pi^e(t)$

$$\pi = a - bN_u + h\pi^e, \quad (0 < h \leq 1), \tag{5.1.1}$$

where a, b, and h are parameters. The adaptive expectations hypothesis establishes a rule of the expected rate of inflation as follows

$$\dot{\pi}^e = j(\pi - \pi^e), \quad 0 < j \leq 1, \tag{5.1.2}$$

which states that if the actual rate of inflation exceeds the expected rate of inflation, then the expected rate of inflation tends to rise.

Denote the nominal money balance by M and its rate of growth by $\mu = \dot{M}/M$. The model contains a feedback from inflation to unemployment

$$\dot{N}_u = -\kappa(\mu - p), \quad \kappa > 0, \tag{5.1.3}$$

where $\mu - \pi$ is the rate of growth of real-money balance and κ is a parameter. The model consists of three equations, (5.1.1), (5.1.2), and (5.1.3), with three variables, π, N_u, and π^e. We now show that the dynamics can be described by a second-order linear differential equation. First, substitute equation (5.1.1) into equation (5.1.2)

$$\dot{\pi}^e = j(a - bN_u) + j(h-1)\pi^e.$$

Taking the above equation with regard to t, we have

$$\ddot{\pi}^e = -jb\dot{N}_u + j(h-1)\dot{\pi}^e.$$

Substituting equation (5.1.1) into this equation results in

$$\ddot{\pi}^e + (b\kappa + j - jh)\dot{\pi}^e + jb\kappa\pi^e = jb\kappa\mu, \tag{5.1.4}$$

where we use $\pi = \dot{\pi}^e/j + \pi^e$. We can explicitly solve equation (5.1.4). As the equation is explicitly solved, we will not discuss different possible combinations of parameter values. Once we solve $\pi(t)$, we determine $\pi = \dot{\pi}^e/j + \pi^e$ and $N_u = (a - \pi + h\pi^e)/b$. We depict an interaction between inflation and unemployment with specified values of the parameters.[3]

5.2 The IS-LM model for a national economy

This section explains the IS-LM model, which was developed by Hicks (1937).[4] This model is often identified with the Keynesian approach to macroeconomics. Since Hicks, many economists have worked to refine the IS-LM model.[5] In the standard IS-LM, wages and prices are fixed at least in the short term. To classical economists who hold that wages and prices move rapidly to clear markets, the assumption of a fixed price is not acceptable.[6] Nevertheless, as the original IS-LM can also be modified to allow rapid adjustment of prices and wage,

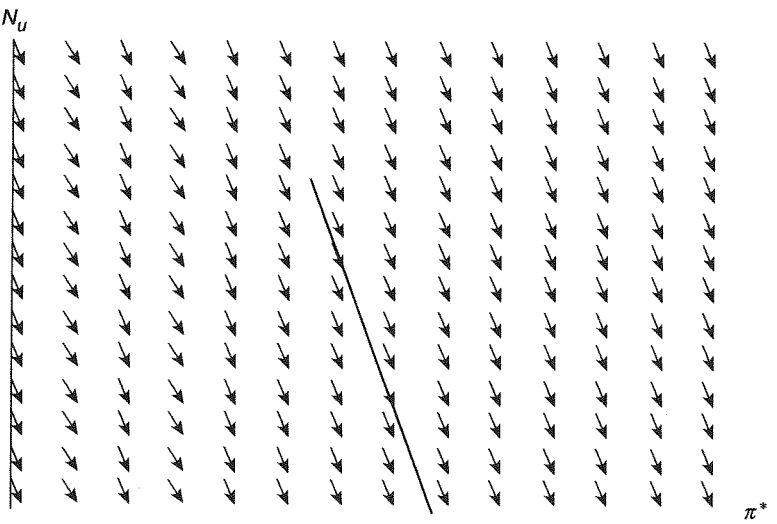


Figure 5.1.1 An interaction between inflation and unemployment.

the analytical framework is now used to analyze classical economic problems as well.

Labor market

With capital stock fixed, we specify a production function as follows

$$F = A\left(\theta N - \frac{\theta_1}{2}N^2\right), \tag{5.2.1}$$

in which F and N are respectively output and labor input, and A, θ, and θ_1 are positive parameters. The marginal condition is given by

$$w = A(\theta - \theta_1 N), \tag{5.2.2}$$

where w is the real wage. This is the demand function for labor. Assume that the labor supply function is specified as follows

$$N = n_0 + n_w(1 - \tau_w)w, \tag{5.2.3}$$

in which n_0 and n_w are positive parameters and τ_w $(0 \leq \tau_w < 1)$ is the tax rate on wage income. When labor demanded is equal to labor supplied, we have

$$N = \frac{n_0 + \theta A n_w(1 - \tau_w)}{1 + n_w\theta_1 A(1 - \tau_w)}. \tag{5.2.4}$$

This is the full-employment level of employment.

Goods market

Assume that the desired consumption is given as follows

$$C = \xi_0 + \xi_Y(Y - T) - \xi_r r, \tag{5.2.5}$$

where Y, T, and r are respectively income, taxes, and the real interest rate, $Y - T$ is disposable income, and ξ_0, ξ_Y, and ξ_r are positive parameters. Here, ξ_Y is the marginal propensity to consume. As people consume some part of an increase in disposable income and save the rest, it is reasonable to require $0 < \xi_Y < 1$. An increase in the real interest rate causes desired consumption to fall.

Taxes are given as

$$T = \tau_0 + \tau_w Y, \tag{5.2.6}$$

where τ_0 is a lump-sum tax and τ_w is the tax rate on income (which is the same as levied on wages). Desired investment is given as

$$I = i_0 - i_r r, \tag{5.2.7}$$

where i_0 and i_r are positive numbers. This equation means that desired investment falls when the real interest rises.

The goods market equilibrium condition is

$$Y = C + I + G,$$

where G is the government's expenditure. Substituting equations (5.2.5)–(5.2.7), we have

$$r = \alpha_{IS} - \beta_{IS} Y, \tag{5.2.8}$$

in which

$$\alpha_{IS} \equiv \frac{\xi_0 + i_0 + G - \xi_Y \tau_0}{\xi_r + i_r}, \quad \beta_{IS} \equiv \frac{1 - (1 - \tau)\xi_Y}{\xi_r + i_r} > 0.$$

Equation (5.2.8) relates output to the real interest rate that clears the goods market. This relationship between Y and r defines the IS curve. The IS curve slopes downward.

The asset market

It is assumed that the real demand for money depends on real income, Y, and the nominal interest rate on nonmonetary assets, i, which in turn equals the expected real interest rate, r, plus the expected rate of inflation, π^e. It is assumed that the money demand function takes the form

$$\frac{M}{P} = \lambda_0 + \lambda_Y Y - \lambda_r (r + \pi^e), \tag{5.2.9}$$

where M is the nominal demand for money and P is the price level, and λ_0, λ_Y, and λ_r are positive parameters. For given levels of M, P, and π^e, equation (5.2.9) relates output to the real interest rate that clears the asset market. This equation defines the LM curve. We rewrite (5.2.9) as follows

$$r = \alpha_{LM} - \frac{M}{\lambda_r P} + \beta_{LM} Y, \tag{5.2.10}$$

where

$$\alpha_{LM} \equiv \frac{\lambda_0}{\lambda_r} - \pi^e, \quad \beta_{LM} \equiv \frac{\lambda_Y}{\lambda_r} > 0.$$

The LM curve slopes upward.

General equilibrium

We determine the general equilibrium levels of the real wage, w, employment, N, and output, Y, by equations (5.2.1), (5.2.2), and (5.2.4). We then determine r by

equation (5.2.8). We then use equations (5.2.5), (5.2.6), and (5.2.7) to determine C, T, and I. By equation (5.2.10), we solve price as

$$P = \frac{M}{\lambda_r(\alpha_{LM} - r + \beta_{LM} Y)}.$$

It is straightforward to examine effects of changes in the parameters upon the economic equilibrium.

The AD-AS model

We have illustrated the IS-LM model. Another important model in macroeconomics is the AD-AS model. The output demanded at any price level is the amount of output determined at the intersection of the IS and LM curves. The output level is given from equations (5.2.8) and (5.2.10) as

$$Y = \frac{\alpha_{IS} - \alpha_{LM} + M/\lambda_r P}{\beta_{IS} + \beta_{LM}}. \tag{5.2.11}$$

This is the aggregate demand (AD) curve. For constant M, Y is negatively related to P.

In the short run, the price level is fixed. The short-run aggregate supply (SRAS) curve is a horizontal line given by $P = \overline{P}$.[7] The short-run equilibrium is represented by the intersection of the AD curve and the SRAS curve.

In the long run, the long-run aggregate supply (LRAS) curve is determined at the full-employment level of output, $Y = \overline{Y}$. The long-run equilibrium of the economy is represented by the intersection of the AD curve and the LSAS curve.

5.3 A monetary growth welfare model with the MIUF approach

We are now concerned regarding a possibility of unemployment that occurs in a welfare economy. The idea is that if society offers a generous welfare for unemployment, people have incentives to "remain unemployed" – earning welfare payments and enjoying leisure. Marx and Keynes proposed different reasons for the existence of unemployment. Neither Marx nor Keynes was concerned with unemployment issues in welfare economies in which an unemployed person may be paid "well." In a generous social system, to work and to be at leisure means two different life styles, which a person may like to choose. This section studies whether or not "natural" unemployment exists in a welfare economic system when the government pays well for the unemployed. We construct a one-sector and one-group growth economic model to show that the natural unemployment may exist under competitive conditions with government intervention in labor markets.

To make unemployment less painful, society may adapt different social welfare policies. Although these adjustments can alleviate some of the burden

Table 5.3.1 Unemployment benefits as a percentage of the wage.

Country	*1960s*	*1970s*	*1980s*	*1991*
France	25	24	34	37
Germany	32	30	29	28
United Kingdom	28	26	24	20
United States	10	13	15	12

of unemployment, some adjustments may actually accelerate unemployment. Table 5.3.1 illustrates the change of unemployment benefits in a number of countries since the 1960s.[8] Unemployment benefits include many terms, such as eligibility conditions to the length of benefit period. The index in the table takes these different dimensions into account by computing the ratio of unemployment benefits to the wages for workers with differential initial wages and different durations of unemployment, and then computing the average ratio over all these cases. We see that the United States was the least generous among the countries cited. In France, unemployment benefits actually became more generous in the 1980s. In some welfare economies, like Sweden and Norway, unemployment benefits were much higher in the 1990s than these cited. It was argued that in France, for instance, the increased generosity had "encouraged" unemployment.

5.3.1 The model with welfare and unemployment

We are now concerned with an economic system similar to the OSG model with unemployment developed in Zhang (2005a: Chapter 7). Let the population be fixed and denoted by N. Let subscript indexes, i and u, denote respectively employed and unemployed. We denote $N_i(t)$ and $N_u(t)$ respectively the number of workers employed by the production sector and the number of persons unemployed. We have

$$N_i(t)+N_u(t)=N.$$

The production function of the economy is specified as

$$F(t)=K^{\alpha}(t)(zN_i(t))^{\beta}, \quad \alpha+\beta=1,$$

where z is the level of human capital, $K(t)$ is the capital stocks of the economy and $F(t)$ is the total output at time t. We assume that z is fixed. The marginal conditions are given by

$$r(t)+\delta_k=\frac{\alpha F(t)}{K(t)}, \quad w_i(t)=\frac{\beta F(t)}{N_i(t)}, \tag{5.3.1}$$

where $r(t)$ is the rate of interest, $w_i(t)$ is the wage rate.

Money is introduced by assuming that a central bank distributes at no cost to the population a per capita amount of fiat money $M(t) > 0$. The scheme according to which the money stock evolves over time is deterministic and known to all agents. With μ being the constant net growth rate of the money stock, $M(t)$ evolves over time according to $\dot{M}(t) = \mu M(t), \mu > 0$. The government expenditure in real terms per capita, $g(t)$, is given by

$$g(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t),$$

where $P(t)$ is the price of the output good and $m(t)$ stands for the real value of money per capita measured in units of the output good. The government will use the income, gN, to pay for the unemployed. The inflation rate, $\pi(t)$, is given by

$$\pi(t) \equiv \frac{\dot{P}(t)}{P(t)} = \mu - \frac{\dot{m}(t)}{m(t)}. \tag{5.3.2}$$

We assume that an unemployed person gets some payment or unemployment compensation rate from the government. How the unemployment payment is determined depends on many factors such as the power of labor union and actual economic conditions. Let each unemployed person be paid $w_u(t)$ amount of money by the government at time t. Household j's disposable income in real terms, $\hat{y}_j(t)$, is

$$\hat{y}_j(t) = r(t)k_j(t) + w_j(t) - \pi(t)m_j(t) + a_j(t), \tag{5.3.3}$$

where $a_j(t) \equiv k_j(t) + m_j(t)$. We assume that at each point of time consumers' preferences over money, consumption and saving can be represented by the following utility function

$$U_j(t) = \theta_j m_j^{\varepsilon_0}(t) c_j^{\xi_0}(t) s_j^{\lambda_0}(t), \quad \varepsilon_0,\ \xi_0,\ \lambda_0 > 0,$$

where ε_0 is called household j's propensity to hold money, ξ_0 propensity to consume, and λ_0 propensity to own wealth. Here, θ_j is the amenity level of person j. We assume θ_j to be constant. This assumption is strict. For instance, it is reasonable to assume that an unemployed person's amenity level, θ_u, is affected by the difference of the wage rate and unemployment payment rate, $w_i(t) - w_u(t)$. Here, we assume that people have two alternatives – either work or not work. It is possible that length of work time is an endogenous variable.

The budget constraint is given by

$$(1 + r(t))m_j(t) + c_j(t) + s_j(t) = \hat{y}_j(t).$$

Insert equation (5.3.4) in the budget constraints

$$(r + \pi)m_j + c_j + s_j = \bar{y}_j \equiv (1 + r)k_j + w_j. \tag{5.3.4}$$

Consumers' problem is to choose current money, consumption, and savings in such a way that utility levels are maximized. Maximizing $U_j(t)$ subject to the budget constraint (5.3.4) yields

$$(r+\pi)m_j = \varepsilon \bar{y}_j, \quad c_j = \xi \bar{y}_j, \quad s_j = \lambda \bar{y}_j. \tag{5.3.5}$$

where

$$\varepsilon \equiv \rho \varepsilon_0, \quad \xi \equiv \rho \xi_0, \quad \lambda \equiv \rho \lambda_0, \quad \rho \equiv \frac{1}{\varepsilon_0 + \xi_0 + \lambda_0}.$$

It is assumed that the population is homogenous and there is no discrimination in labor market. We also assume that labor market adapts smoothly and rapidly. It is possible that people would choose this life style when consumption in association with it is not very low. Labor market is assumed to function in such a way that people stop changing their current situation when their utility levels are the same for either of the two types of life, i.e.

$$U_u(t) = U_i(t), \quad \text{if } N_u(t) > 0.$$

Here, we are only interested in the conditions that "unemployment" exists in the system.

The real wealth changes as follows

$$\dot{a}_j(t) = s_j(t) - a_j(t). \tag{5.3.6}$$

The total demand for money is equal to the total supply

$$m_i(t)N_i(t) + m_u(t)N_u(t) = Nm(t). \tag{5.3.7}$$

The assumption of full employment of capital is given by

$$K(t) = k_i(t)N_i(t) + k_u(t)N_u(t),$$

that is

$$k(t) = n_i k_i(t) + n_u k_u(t), \tag{5.3.8}$$

where $n_j(t) \equiv N_j(t)/N$.

The government's expenditure is only for welfare of the unemployed. We neglect the government's other possible expenditures. The government's budget constraint is

$$w_u(t)N_u(t) = G(t) \equiv \mu m(t)N. \tag{5.3.9}$$

We assume that the government balances its income at any point of time. The government may intervene labor market in different ways. In this model, we assume

that the government fixes μ. It should be noted that this model is not realistic in the sense that the condition (5.3.9) with endogenous w_u and $U_i = U_u$ "guarantees" the existence of unemployment. As we are concerned with modeling the force of social welfare for existence of unemployment in the monetary economy, for illustration we accept this condition.[9] The model has been completed. We now examine its properties.

5.3.2 The motion of the system

We now try to find differential equations governing the dynamic system. First, multiplying equation $(r+\pi)m_j = \varepsilon \bar{y}_j$ in (5.3.5) by N_j and then adding up the result of the two equations, we obtain

$$r+\pi = \frac{\varepsilon(N_i\bar{y}_i + N_u\bar{y}_u)}{m_1N_i + m_2N_u}.$$

Substituting equation (5.3.8) into the above equation yields

$$\pi = \frac{(\bar{y}_i + u\bar{y}_u)\varepsilon N_i}{m} - r, \tag{5.3.10}$$

where $u(t) \equiv N_u(t)/N_i(t)$. Insert equation (5.3.3) into equation (5.3.10)

$$\dot{m} = (r+\mu)m - (\bar{y}_i + u\bar{y}_u)\varepsilon N_i. \tag{5.3.11}$$

Substituting equations (5.3.5) into $U_u = U_i$ yields $\bar{y}_i = \theta\bar{y}_u$, where $\theta \equiv (\theta_u/\theta_i)^\rho$. Substituting $\bar{y}_j = (1+r)k_j + w_j$ into this equation yields

$$(1+r)(k_i - \theta k_u) + w_i = \frac{\theta\mu mN}{N_u},$$

where we also use $w_uN_u = \mu mN$. Substituting

$$r = \frac{\alpha z^\beta}{(k_i+uk_u)^\beta} - \delta_k, \quad w_i = (k_i+uk_u)^\alpha \beta z^\beta, \quad N_u = \frac{uN}{1+u}, \tag{5.3.12}$$

into the above equation, we have

$$\begin{aligned} m &= \tilde{\Lambda}(k_i, k_u, u) \\ &\equiv \left[\left(1 - \delta_k + \frac{\alpha z^\beta}{(k_i+uk_u)^\beta}\right)(k_i - \theta k_u) + (k_i+uk_u)^\alpha \beta z^\beta\right]\frac{u}{\theta\mu(1+u)}. \end{aligned} \tag{5.3.13}$$

Taking derivatives of this equation with regard to t yields

$$\dot{m} = \tilde{\Lambda}_{k_i}\dot{k}_i + \tilde{\Lambda}_{k_u}\dot{k}_u + \tilde{\Lambda}_u\dot{u}.$$

From equations (5.3.12) and the definitions of $\bar{y}_j$, we can now consider r, w_j, m, and $\bar{y}_j$ as functions of k_i, k_u, and u. From equations (5.3.11) and the above equation, we solve

$$\tilde{\Lambda}_{k_i}\dot{k}_i + \tilde{\Lambda}_{k_u}\dot{k}_u + \tilde{\Lambda}_u\dot{u} = (r+\mu)\tilde{\Lambda} - (\bar{y}_i + u\bar{y}_u)\varepsilon N_i. \tag{5.3.14}$$

From equation (5.3.10) and $\pi = \mu - \dot{m}/m$, we have

$$\pi = \frac{(\bar{y}_i + u\bar{y}_u)\varepsilon N}{(1+u)\tilde{\Lambda}} - r. \tag{5.3.15}$$

Insert this equation in $(r+\pi)m_j = \varepsilon\bar{y}_j$

$$m_j = \Lambda_j(k_i, k_u, u) \equiv \frac{(1+u)\varepsilon\bar{y}_j\tilde{\Lambda}}{(\bar{y}_i + u\bar{y}_u)\varepsilon N}. \tag{5.3.16}$$

Taking derivatives of equations (5.3.16) with respect to t yields

$$\dot{m}_j = \frac{\partial\Lambda_j}{\partial k_i}\dot{k}_i + \frac{\partial\Lambda_j}{\partial k_u}\dot{k}_u + \frac{\partial\Lambda_j}{\partial u}\dot{u}, \quad j = i, u. \tag{5.3.17}$$

Substituting $s_j = \lambda\bar{y}_j$ and $a_j = k_j + m_j$ into equations (5.3.6) yields

$$\left(1 + \frac{\partial\Lambda_i}{\partial k_i}\right)\dot{k}_i + \frac{\partial\Lambda_i}{\partial k_u}\dot{k}_u + \frac{\partial\Lambda_i}{\partial u}\dot{u} = \lambda y_i - k_i - \Lambda_i,$$

$$\frac{\partial\Lambda_u}{\partial k_i}\dot{k}_i + \left(1 + \frac{\partial\Lambda_u}{\partial k_u}\right)\dot{k}_u + \frac{\partial\Lambda_u}{\partial u}\dot{u} = \lambda y_u - k_u - \Lambda_u. \tag{5.3.18}$$

We can write equations (5.3.14) and (5.3.18) in the matrix form

$$\begin{bmatrix} 1+\dfrac{\partial\Lambda_i}{\partial k_i} & \dfrac{\partial\Lambda_i}{\partial k_u} & \dfrac{\partial\Lambda_i}{\partial u} \\ \dfrac{\partial\Lambda_u}{\partial k_i} & 1+\dfrac{\partial\Lambda_u}{\partial k_u} & \dfrac{\partial\Lambda_u}{\partial u} \\ \tilde{\Lambda}_{k_i} & \tilde{\Lambda}_{k_u} & \tilde{\Lambda}_u \end{bmatrix} \begin{bmatrix} \dot{k}_i \\ \dot{k}_u \\ \dot{u} \end{bmatrix} = \begin{bmatrix} \lambda y_i - k_i - \Lambda_i \\ \lambda y_u - k_u - \Lambda_u \\ (r+\mu)\tilde{\Lambda} - (\bar{y}_i + u\bar{y}_u)\varepsilon N_i \end{bmatrix}. \tag{5.3.19}$$

Lemma 5.3.1[10]

The motion of $k_i(t), k_u(t)$, and $u(t)$ is determined by the three differential equations (5.3.19). For any positive $k_i(t), k_u(t)$, and $u(t)$, all the other variables are uniquely determined at any point of time by the following procedure: r, w_i,

and N_u by (5.3.12) $\rightarrow N_i = N - N_u \rightarrow K = k_i N_i + k_u N_u \rightarrow F = K^\alpha (zN_i)^\beta \rightarrow m$ by (5.3.13) $\rightarrow m_j, j = i, u$, by (5.3.16) $\rightarrow \bar{y}_j = (1+r)k_j + w_j \rightarrow \pi$ by (5.3.15) $\rightarrow m_j, c_j$, and s_j by (5.3.5) $\rightarrow a_j = k_j + m_j$.

The lemma is important for that it provides a procedure for computing the motion of the entire system. For any proper initial conditions, it is straightforward to simulate the model with computer. As the expressions are tedious, we do not discuss under what conditions the dynamic system has meaningful solutions. We now study steady states.

5.3.3 The existence of steady state

At steady state we have $\pi = \mu$ and $\lambda \bar{y}_j = a_j$. From $\lambda \bar{y}_j = a_j$ and $a_j = k_j + m_j$, we have

$$m_j = (\lambda - 1 + \lambda r)k_j + \lambda w_j, \tag{5.3.20}$$

where we use $\bar{y}_j = (1+r)k_j + w_j$. From $\pi = \mu$ and $(r+\pi)m_j = \varepsilon \bar{y}_j$, we have $(r+\mu)m_j = \varepsilon \bar{y}_j$. Insert $\bar{y}_j = (1+r)k_j + w_j$ into these equations

$$(r+\mu)m_j = (1+r)\varepsilon k_j + \varepsilon w_j. \tag{5.3.21}$$

Deleting m_j from equations (5.3.20) and (5.3.21), we have

$$[(\lambda - 1 + \lambda r)(r+\mu) - (1+r)\varepsilon]k_j = (\varepsilon - \lambda r - \lambda\mu)w_j, \quad j = i, u. \tag{5.3.22}$$

Dividing the above two equations, we obtain $w_u = k_u w_i / k_i$. From the above equation, $\bar{y}_i = \theta \bar{y}_u$ and $\bar{y}_j = (1+r)k_j + w_j$, we have

$$k_u = \frac{k_i}{\theta}, \quad w_u = \frac{w_i}{\theta}. \tag{5.3.23}$$

From equation (5.3.9) and equation (5.3.24), we have $\beta u F = \theta \mu m N$, where we also use $w_i = \beta F / N_i$ and $u \equiv N_u / N_i$. Substituting $F = K^\alpha (zN_i)^\beta$ into $\beta u F = \theta \mu m N$ yields

$$k_i^\alpha u \left(1 + \frac{u}{\theta}\right)^\alpha = z_0 m (1+u), \tag{5.3.24}$$

where we use $K = (k_i + u k_u)N_i$ and $z_0 \equiv \theta\mu / \beta z^\beta$. From $\bar{y}_i = \theta \bar{y}_u$ and equations (5.3.5), we have $m_u = m_i / \theta$. From $m_i N_i + m_u N_u = Nm$ and $m_u = m_i / \theta$, we have

$$m_i = \frac{Nm}{N_i + N_u / \theta}. \tag{5.3.25}$$

Multiplying $(r+\mu)m_j = \varepsilon \bar{y}_j$ by N_j and then adding the two resulted equations, we have

$$\left(r+\mu-\frac{\varepsilon}{\lambda}\right)\frac{Nm}{1+u/\theta} = \frac{\varepsilon k_i N_i}{\lambda}, \tag{5.3.26}$$

where we use $\bar{y}_i = a_i/\lambda$ and equation (5.3.25). From equations (5.3.24) and (5.3.26), we delete m

$$r+\mu-\frac{\varepsilon}{\lambda} = \left(1+\frac{u}{\theta}\right)^{\beta}\frac{\varepsilon z_0 k_i^{\beta}}{\lambda u}.$$

Substitute $r = \alpha z^{\beta} k_i^{-\beta}(1+u/\theta)^{-\beta} - \delta_k$ into the above equation

$$\left(1+\frac{u}{\theta}\right)^{2\beta} k_i^{2\beta} + \left(1+\frac{u}{\theta}\right)^{\beta}\varepsilon_1 u k_i^{\beta} - \varepsilon_0 u = 0,$$

where

$$\varepsilon_1 \equiv \left(\frac{\varepsilon+\lambda\delta_k-\mu\lambda}{\varepsilon z_0}\right), \quad \varepsilon_0 \equiv \frac{\lambda\alpha z^{\beta}}{\varepsilon z_0} > 0.$$

It is reasonable to require $\varepsilon+\lambda\delta_k > \mu\lambda$, which implies $\varepsilon_1 > 0$. Solve the above equation

$$k_i = \frac{\left(-\varepsilon_1 u + \sqrt{\varepsilon_1^2 u^2 + 4\varepsilon_0 u}\right)^{1/\beta}}{2^{1/\beta}(1+u/\theta)}. \tag{5.3.27}$$

Multiplying the equation for m_j in (5.3.20) by N_j and adding the two resulted equations, we have

$$\frac{u}{z_0}\left(1+\frac{u}{\theta}\right)^{-\beta} = \left(\alpha\lambda z^{\beta}(1+u/\theta)^{-\beta} - (1-\lambda+\lambda\delta_k)k_i^{\beta}\right) + \left(1+\frac{u}{\theta}\right)^{\alpha}\lambda\beta z^{\beta},$$

where we use $r = \alpha z^{\beta} k_i^{-\beta}(1+u/\theta)^{-\beta} - \delta_k$, $w_i = (1+u/\theta)^{\alpha}\beta z^{\beta} k_i^{\alpha}$ and equations (5.3.7), (5.3.23), and (5.3.24). Insert equation (5.3.27) in the above equation

$$\sqrt{\varepsilon_1^2 u^2 + 4\varepsilon_0 u} = \lambda\delta z^{\beta} + \bar{\varepsilon} u,$$

where

$$\delta \equiv \frac{2}{1-\lambda+\lambda\delta_k} > 0, \quad \bar{\varepsilon} \equiv \varepsilon_1 - \left(\frac{1-\lambda\mu}{z_0}\right)\delta = \frac{\varepsilon+\lambda\delta_k-\mu\lambda-\varepsilon\delta+\varepsilon\delta\lambda\mu}{\varepsilon z_0}.$$

We rewrite the above equation as

$$u^2 + \delta_0 u + \delta_1 = 0, \tag{5.3.28}$$

where

$$\delta_0 \equiv 2\delta_2\left(\lambda\delta z^{\beta}\bar{\varepsilon} - 2\varepsilon_0\right), \quad \delta_1 \equiv \delta_2\left(\lambda\delta z^{\beta}\right)^2,$$

$$\delta_2 \equiv \left\{\left[\left(\frac{1-\lambda\mu}{z_0}\right)\delta - 2\varepsilon_1\right]\left(\frac{1-\lambda\mu}{z_0}\right)\delta\right\}^{-1}.$$

From the definitions of the parameters, we have

$$\lambda\delta z^{\beta}\bar{\varepsilon} - 2\varepsilon_0 = \left[(\beta-\delta)\varepsilon - (1-\varepsilon\delta)\mu\lambda - \alpha\xi + \beta\lambda\delta_k\right]\frac{\delta\lambda z^{\beta}}{\varepsilon z_0},$$

where $\delta\lambda z^{\beta}/\varepsilon z_0 > 0$. It can be seen that if ε and δ_k are small, then the term, $\lambda\delta z^{\beta}\bar{\varepsilon} - 2\varepsilon_0$, is negative. In the rest of this section, we require this term to be negative.[11] From the definitions of δ_0 and δ_1, the signs of δ_0 and δ_1 are opposite. This guarantees that equation (5.3.28) has a unique positive solution[12]

$$u = \frac{-\delta_0 + \sqrt{\delta_0^2 - 4\delta_1}}{2}. \tag{5.3.29}$$

From the definition of u and $N_i + N_u = N$, we determine the labor distribution as follows

$$N_i = \frac{N}{1+u}, \quad N_u = \frac{uN}{1+u}. \tag{5.3.30}$$

Lemma 5.3.2

If ε and δ_k are sufficiently small, then the dynamic system has a unique steady state. The variables are given by the following procedure: u by (5.3.29) $\rightarrow N_i$ and N_u by (5.3.30) $\rightarrow k_i$ by (5.3.27) $\rightarrow k_u = k_i/\theta \rightarrow w_i = (1+u/\theta)^{\alpha}\beta z^{\beta}k_i^{\alpha} \rightarrow w_u = w_i/\theta \rightarrow r = \alpha z^{\beta}k_i^{-\beta}(1+u/\theta)^{-\beta} - \delta_k \rightarrow K = k_iN_i + k_uN_u \rightarrow F = K^{\alpha}(zN_i)^{\beta} \rightarrow m_j$, $j = i, u$, by (5.3.20) $\rightarrow m = \beta uF/\theta\mu N \rightarrow \bar{y}_j = (r+\mu)m_j/\varepsilon \rightarrow c_j$ and s_j by (5.3.5) $\rightarrow a_j = k_j + m_j$.

As we explicitly solved the steady state, it is straightforward to examine effects of changes in any parameter on the economy. For instance, we may examine effects of changes in the inflation policy parameter, μ, on the economy. Taking derivatives of equation (5.3.28) with respect to μ yields

$$(2u+\delta_0)\frac{du}{d\mu} = -u\frac{d\delta_0}{d\mu} - \left[\frac{(1-\lambda\mu)(\delta - 1/\varepsilon) - z_0\varepsilon_1}{(1-\lambda\mu)\left[(1-\lambda\mu)\delta - 2z_0\varepsilon_1\right]}\right]2\lambda\delta_1,$$

where $2u+\delta_0=\sqrt{\delta_0^2-4\delta_1}>0$ and

$$\frac{d\delta_0}{d\mu}=\lambda\delta_0\left[\frac{(1-\lambda\mu)2\delta-2z_0\varepsilon_1-(1-\lambda\mu)2/\varepsilon}{(1-\lambda\mu)\left[(1-\lambda\mu)\delta-2z_0\varepsilon_1\right]}\right]+\left(\delta-\frac{1}{\varepsilon}\right)\frac{2\delta_2\lambda^2\delta z^\beta}{z_0}.$$

We will not further examine the problem because it is not easy to explicitly interpret the economic implications of the analysis.

5.4 Money, the efficiency–wage theory and the MIUF approach

The phenomenon of involuntary unemployment poses a challenge for conventional economic theory. If the Walrasian labor market works smoothly, any excess supply of labor ought to be eliminated by wage cuts. According to the traditional neoclassical economic theory, a profit-maximizing employer would accept an unemployed worker's offer of the same labor services but asking for less pay than the going wage. Efficiency–wage theories identify some rationales that such arbitrage ought to be prevented. The key concept of the theories is that if the productive efficiency of the worker depends on the wage, a wage cut will be accompanied by a drop in the worker's efficiency, thus rendering the arbitrage worthless to the employer.

The development of efficiency–wage theories was independently initiated by Leibenstein (1957) and Mazumdar (1959). The early idea is to build the link between nutrition intake and work efficiency and to explore the effects of this link on wages and involuntary unemployment. If a worker is paid too low and his/her effort is dependent on the income, the employer would not be interested in hiring him with a low payment because of low productivity. This idea for poor economies has found its interpretations of involuntary unemployment in rich economies. This idea also shows that if the price of a production factor has functions rather than the conventional market-clearing one, one may get beyond the confines of the market-clearing Walrasian equilibrium.

One may ask why firms turn away qualified job applicants who are willing to work for less than the firm's current wage offer. To explain this, let us denote e and w respectively an effort level of workers and a wage rate offered by the firm. It is assumed that worker effort and productivity are positive associated with the wages paid by the firm, $e=e(w)$. We also assume that the effort function is concave; that is, $de/dw>0$ and $d^2e/dw^2<0$. For the firm, we consider a production function with the neoclassical properties: $F=F(K,N^*)$, where N^* is the number N^d of labor demanded times the effort level per worker. The production function can then be written as: $F=F\left(K,e(w)N^d\right)$. Profits Π are

$$\Pi=F\left(K,e(w)N^d\right)-rK-wN^d.$$

Since the firm's profits depend on the capital stock, the number of workers and on the wage rate (which determine the effort), we must take first-order conditions

with respect to three choice variables. These first-order conditions are

$$\Pi_K = F_K - r = 0,$$

$$\Pi_{N^d} = F_{N^*} e - w = 0,$$

$$\Pi_w = F_{N^*} N^d e' - N^d = 0.$$

From $F_{N^*} e = w$ and $F_{N^*} e' = 1$, we find that

$$\frac{de}{dw}\frac{w}{e} = 1.$$

There is a direct and increasing relation between the wage paid by firms and the effort provided by workers. Maximizing profits set an efficiency wage such that the effort–wage elasticity is equal to unity. This is referred to as the Solow condition (Solow, 1979). This is the well-known conclusion of the theory: the firm's profit maximizing wage rate must therefore be chosen so that the effort function is unit elastic with respect to changes in the wage rate. This condition is independent of the total factor productivity and the level of capital input. Since the optimal wage rate is not determined by balance of demand and supply in labor market, profit-maximizing microeconomic choices do not guarantee macroeconomic full employment.

The efficiency–wage theory is obtained in a model with identical, perfectly competitive firms. Graphically, the relation between the real wage and the level of effort is shown by the effort curve as in Figure 5.4.1a. The effort curve passes through the origin. When real wages are higher, workers choose to work harder – the effort curve slopes upward. The figure gives a S-shaped effort curve. At the lowest levels of the wage, workers make hardly any effort, and effort rises only slowly as the real wage increases. At higher levels of wage effort rises sharply. The curve flattens at very high levels of the wage because there is some maximum level of effort that workers really cannot exceed. At any point on the curve, effort per dollar of wage is e/w, which equals the slope of the line from the origin to the point. It is reasonable to assume that firms will choose the level of the wage that gets the most effort from workers for each dollar of wages paid. The highest level of effort per dollar of wage is at (w^*, e^*), where the line from the origin is tangent to the curve. That is, $de/dw = e/w$ at (w^*, e^*). As shown in Figure 5.4.1a, the slope of the line from the origin to the point (w^*, e^*) is greater than the slope of the line from the origin to any other point on the effort curve. The wage that maximizes effort per dollar of wages is the efficiency wage.

In efficiency–wage theories, it is assumed that there is a cost as well as a benefit to firms of paying lower wages. It is possible to harm a firm if it pays lower wages. Firms would not reduce wages because higher wages may raise the productivity of labor. Firms try to create idiosyncratic value for their workers in their jobs, so that the threat to terminate deters misconduct such as shirking. For instance, an increase in a worker' income can cause him to be better nourished and thus

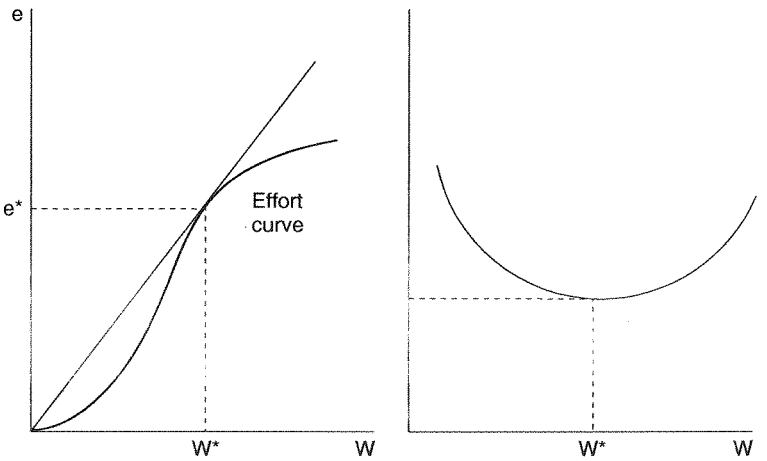


Figure 5.4.1 The efficiency–wage theory.

enables him to work more effectively. A higher wage (than the market-average wage) can increase workers' efforts because the jobs have "additional values" to them and they do not want to lose the jobs. Workers secure jobs through working hard. A high wage can also build loyalty among workers. High loyalty increases the firm's labor productivity – for instance, costs due to strikes can be avoided by high wages. As high wages can also attract workers with high reservation wages, the applicant pool is expanded by high wages. We may explain behavior of firms from the cost perspective. Raising wage rate raises labor cost by making firms pay more to workers, but reduces labor cost per unit of output by making workers more efficient. As shown in Figure 5.4.1b, initially a percent increase in real wage rate raises e by more than 1 percent, so labor cost declines. Firms continue raising w until it reaches the equilibrium value w^*, which occurs at the point where raising w by 1 percent raises efficiency by exactly 1 percent, thus leaving labor cost per unit of efficiency unchanged. Firms will not raise real wage rate any further, since this would raise wage payment more than it would boost efficiency. From how this efficiency wage is determined, we see that firms have no incentive to cut wages when there is unemployment in labor market.

The efficiency–wage theory explains wage rigidity. Because firms choose the wage that maximizes effort received per dollar paid, as long as the effort curve does not change, firms would not change the wages. Employers will pay wage premiums to create the needed idiosyncratic value. Because it is impossible for every employer to pay a premium, there is unemployment in equilibrium. Efficiency–wage models do not suggest that workers have elastic labor supply in the conventional sense. The wage is permanently rigid and equals the efficiency wage.

The model analyzed above is static. It has been pointed out that the condition that the effort–wage elasticity is equal to unity is unrealistic. This condition has been relaxed, for instance, by introducing fixed employment costs per worker in the profit function (thinking of employer-provided health insurance, for instance); by taking into account specific taxes on labor; by considering external turnover costs. Additional costs for employers make the effort–wage elasticity less than one. Jellah and Zenou develop a dynamic efficiency wage model with learning by doing.[13] The article also provides a comprehensive source for studies of efficiency wage models. The model assumes that workers accumulate human capital which allows them to increase their effort. It is shown that if the change in workers' human capital is taken into account, firms set efficiency wages such that the effort-wage elasticity is not in general equal to one.

The workings of the labor market with efficiency wages are shown in Figure 5.4.2. The efficiency wage, w^*, is indicated by a horizontal line. The upward-sloping curve is a standard labor supply curve, showing the number of hours of work that people would like to supply at each level of wage. The downward-sloping curve is the demand curve for labor in the efficiency–wage model. This curve is given by the marginal condition,

$$F_{N^*}e = w,$$

for any given level of K. We now show that the labor demand curve slopes down. Taking derivatives of $F_{N^*} = w/e$ with respect to w yields

$$\left(e'N^d + e\frac{dN^d}{dw}\right)\frac{\partial^2 F}{\partial^2 N^*} = \left(1 - e'\frac{w}{e}\right)\frac{1}{e}, \quad \frac{\partial^2 F}{\partial^2 N^*} \neq 0.$$

We obtain

$$\frac{dN^d}{dw} = -\frac{e'N^d}{e} < 0.$$

It can be shown that an increase in productivity or in K shifts the labor demand curve to the right. In addition, any change in the effort curve that leads to an increase in the effort level raises the marginal product of labor, and the labor demand curve shifts to the right.

In the case of the Walrasian economy, equilibrium occurs at the intersection of labor demand and supply, denoted full employment in Figure 5.4.2. Point A on the labor demand curve indicates that when the wage is fixed at the efficiency wage, firms want to employ N^* units of labor. Point B on the labor supply curve indicates that when the wage is fixed at the efficiency wage, workers want to supply $\overline{N}$ units of labor, which is greater than the amount demanded by firms. The economy has $\overline{N} - N^*$ units of labor unemployed. Unemployment cannot be solved by market mechanism because in a labor market with an efficiency wage the wage cannot be bid down by people offering to work at lower wages because employers would not hire them.

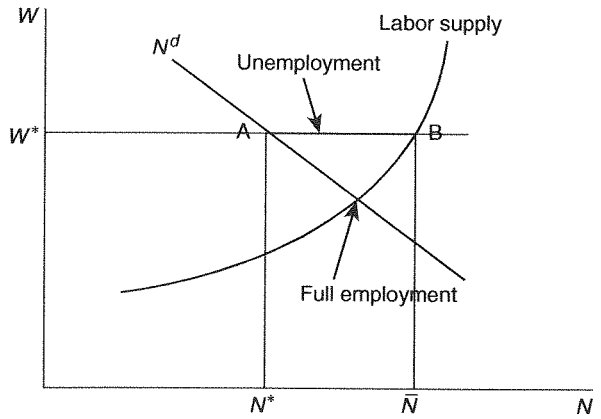


Figure 5.4.2 Labor supply and the efficiency–wage theory.

5.4.1 Unemployment and the wage–efficiency theory in the OSG-MIUF model

We now introduce the wage efficiency theory into the OSG model. Since the optimal wage rate in the wage–efficiency theory is determined not by demand and supply balance in the labor market, it is necessary for this rate to be higher than the wage rate determined by the balance condition of demand and supply in the labor market for existence of unemployment. Once we notice this condition, it is straightforward to provide conditions for generating unemployment in our growth model.

Let us denote e and w_i respectively an effort level of workers and a wage rate offered by the firm. Following the efficiency–wage theory, we assume that effort level is dependent on the real wage rate.[14] This section follows the efficiency–wage theory, assuming

$$e(t) = e(w_i(t)), \quad e'(w_i) > 0, \quad e''(w_i) < 0, \quad \forall w_i(t) \geq 0.$$

Let $N_i(t)$ stand for the employment level determined by the firm. The production function is

$$F(t) = AK^{\alpha}(t)N^{*\beta}(t),$$

where the effective labor input is $N^* = e(w_i)N_i$. The first-order conditions for maximizing profit are

$$r + \delta_k = \frac{\alpha\bar{\tau}F}{K}, \quad w_i = \frac{\beta\bar{\tau}Fe}{N^*}, \quad \frac{\beta\bar{\tau}Fe_{w_i}}{N^*} = 1, \tag{5.4.1}$$

where $\bar{\tau}(t) = 1 - \tau(t)$, in which $\tau(t)$ is the tax rate at time t. The wage rate is uniquely determined by $e_{w_i} = e/w_i$ (independent of market conditions) and the

rate of interest is given for the firm. In the remainder of this section, we treat the wage rate as a "parameter" in the system. As w_i is predetermined, so is $e(w_i)$.

Money is introduced by assuming that a central bank distributes at no cost to the population a per capita amount of fiat money $M(t) > 0$. The scheme according to which the money stock evolves over time is deterministic and known to all agents. With μ being the constant net growth rate of the money stock, $M(t)$ evolves over time according to $\dot{M}(t) = \mu M(t), \mu > 0$. The government expenditure in real terms per capita, $g(t)$, is given by

$$g(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t),$$

where $P(t)$ is the price of the output good and $m(t)$ stand for the real value of money per capita measured in units of the output good. We assume that the government distributes its inflation income, gN, equally among the households. The government pays the unemployed with the tax income. The inflation rate, $\pi(t)$, is given by

$$\pi(t) \equiv \frac{\dot{P}(t)}{P(t)} = \mu - \frac{\dot{m}(t)}{m(t)}. \tag{5.4.2}$$

Household j's disposable income in real terms, $\hat{y}_j(t)$, is

$$\hat{y}_j(t) = r(t)k_j(t) + w_j(t) - \pi(t)m_j(t) + a_j(t) + \mu m(t), \tag{5.4.3}$$

where $a_j(t) \equiv k_j(t) + m_j(t)$. We assume that at each point of time consumers' preferences over money, consumption and saving can be represented by the following utility function

$$U_j(t) = \theta_j m_j^{\varepsilon_0}(t) c_j^{\xi_0}(t) s_j^{\lambda_0}(t), \quad \varepsilon_0, \xi_0, \lambda_0 > 0,$$

where ε_0 is called household j's propensity to hold money, ξ_0 propensity to consume, and λ_0 propensity to own wealth. Here, θ_j is the amenity level of person j. We assume θ_j to be constant. This assumption is strict. For instance, it is reasonable to assume that an unemployed person's amenity level θ_u may be affected by the difference $w_i(t) - w_u(t)$ of the wage rate and unemployment payment rate. The budget constraint is given by

$$(1 + r(t))m_j(t) + c_j(t) + s_j(t) = \hat{y}_j(t).$$

Insert equation (5.4.4) into the budget constraints

$$(r + \pi)m_j + c_j + s_j = \bar{y}_j \equiv (1 + r)k_j + w_j + \mu m. \tag{5.4.4}$$

Consumers' problem is to choose current money, consumption, and savings in such a way that utility levels are maximized. Maximizing $U_j(t)$ subject to the

budget constraint (5.4.4) yields

$$(r+\pi)m_j = \varepsilon \bar{y}_j, \quad c_j = \xi \bar{y}_j, \quad s_j = \lambda \bar{y}_j, \tag{5.4.5}$$

where

$$\varepsilon \equiv \rho \varepsilon_0, \quad \xi \equiv \rho \xi_0, \quad \lambda \equiv \rho \lambda_0, \quad \rho \equiv \frac{1}{\varepsilon_0 + \xi_0 + \lambda_0}.$$

It is assumed that the population is homogenous and there is no discrimination in labor market. We also assume that labor market adapts smoothly and rapidly. It is possible that people would choose this life style when consumption in association with it is not very low. Labor market is assumed to function in such a way that people stop changing their current situation when their utility levels are the same for either of the two types of life, i.e.

$$U_u(t) = U_i(t), \quad \text{if } N_u(t) > 0.$$

Here, we are only interested in the conditions that "unemployment" exists in the system.

The real wealth changes as follows

$$\dot{a}_j(t) = s_j(t) - a_j(t). \tag{5.4.6}$$

The total demand for money is equal to the total supply

$$Nm(t) = m_i(t)N_i(t) + m_u(t)N_u(t). \tag{5.4.7}$$

The assumption of full employment of capital is given by

$$K(t) = k_i(t)N_i(t) + k_u(t)N_u(t),$$

that is

$$k(t) = n_i k_i(t) + n_u k_u(t), \tag{5.4.8}$$

where $n_j(t) \equiv N_j(t)/N$.

The government is responsible only for welfare of the unemployed. We neglect the government's other possible expenditures. The government's budget constraint is

$$w_u(t)N_u(t) = G(t) \equiv \tau(t)F(t). \tag{5.4.9}$$

The left-hand side of the budget constraint is the total amount of money paid to the unemployment and the right-hand side is the government's total payment for unemployment compensation. We assume that the government balances its income at any point of time. As there are so many ways for the government to tax and distribute, this specification is an extreme case. As the tax rate is endogenous in the model, the variable, w_u, is endogenously determined at the level at which the employed and unemployed achieves the same level of utility in the welfare economy. The model has been completed. We now examine its properties.

5.4.2 The motion and steady state

Solve $\beta\bar{\tau}Fe/N^* = w_i$ in K

$$K = \frac{w_0 N_i}{\bar{\tau}^{1/\alpha}}. \tag{5.4.10}$$

where $w_0 \equiv \left(w_i/A\beta e^{\beta}\right)^{1/\alpha}$, which is constant. Insert equation (5.4.10) in equation (5.4.8)

$$\bar{\tau} = \left(\frac{w_0}{k_i + k_u u}\right)^{\alpha}. \tag{5.4.11}$$

where $u(t) \equiv N_u(t)/N_i(t) \geq 0$. From the definition of u and $N_i + N_u = N$, we have

$$N_i = \frac{N}{1+u}, \quad N_u = \frac{uN}{1+u}. \tag{5.4.12}$$

For given level of the tax rate and the wealth levels of the households by equation (5.4.11) the determination of unemployment is illustrated in Figure 5.4.3.

If $u(t) = 0$, the labor force is fully employed. From $\beta\bar{\tau}Fe/N^* = w_i$ and equation (5.4.9), we have

$$w_u = \frac{\tau w_i}{u\bar{\tau}\beta e}. \tag{5.4.13}$$

The unemployment rate, $u_0(t)$, is given by

$$u_0(t) \equiv \frac{N_u(t)}{N} = \frac{u(t)}{1+u(t)}.$$

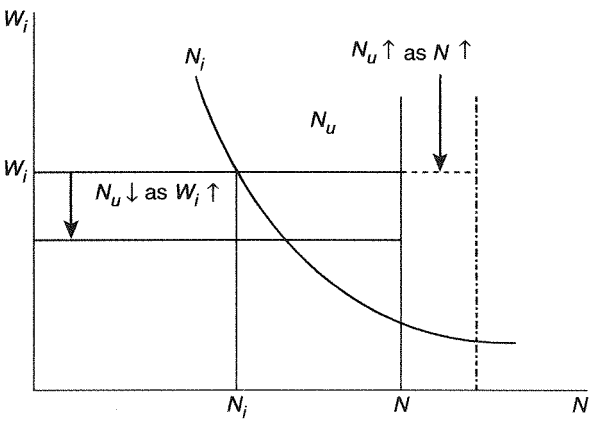


Figure 5.4.3 The efficiency–wage theory in the OSG model.

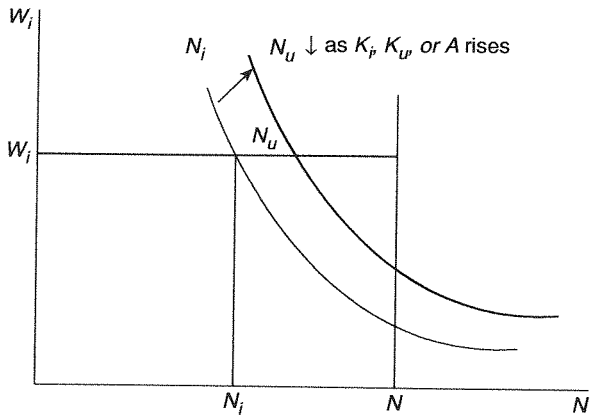


Figure 5.4.4 Unemployment is affected by wealth and productivity.

It is direct to show that $u(t)$ declines in $k_u(t)$, $k_i(t)$, and A. An increase in any people's wealth or productivity tends to increase the unemployment rate. We depict changes in the unemployed as in Figure 5.4.4.

Multiplying equation $(r+\pi)m_j = \varepsilon \bar{y}_j$ in (5.4.5) by N_j and then adding up the resulted two equations, we obtain

$$r+\pi = \frac{\varepsilon(N_i\bar{y}_i + N_u\bar{y}_u)}{m_1N_i + m_2N_u}.$$

Substituting equation (5.4.7) into the above equation yields

$$\pi = \frac{(\bar{y}_i + u\bar{y}_u)\varepsilon}{(1+u)m} - r. \tag{5.4.14}$$

Insert equation (5.4.3) in equation (5.4.14)

$$\dot{m} = (r+\mu)m - \frac{(\bar{y}_i + u\bar{y}_u)\varepsilon}{(1+u)}. \tag{5.4.15}$$

From equations (5.4.1) we obtain

$$r = \frac{\alpha w_i}{(k_i + uk_u)\beta} - \delta_k, \tag{5.4.16}$$

where we use $K = k_iN_i + k_uN_u$. From equations (5.4.11)–(5.4.13), and (5.4.16), we see that r, w_u, N_u, and N_i are functions of u, k_i, and k_u. Substituting the definitions of $\bar{y}_j$ into equation (5.4.15) yields

$$\dot{m} = (r+\mu-\mu\varepsilon)m - \frac{(1+r)(k_i+uk_u) + w_i + uw_u}{(1+u)}. \tag{5.4.17}$$

Substituting equations (5.4.5) into $U_u = U_i$ yields

$$\bar{y}_i = \theta \bar{y}_u,$$

where $\theta \equiv (\theta_u/\theta_i)^{\rho}$. Inserting $\bar{y}_j = (1+r)k_j + w_j + \mu m$ into $\bar{y}_i = \theta \bar{y}_u$, we get

$$m = \tilde{\Lambda}(k_i, k_u, u) \equiv \frac{(1+r)(\theta k_u - k_i) + \theta w_u - w_i}{(1-\theta)\mu}. \tag{5.4.18}$$

Taking derivatives of this equation with regard to t yields

$$\dot{m} = \tilde{\Lambda}_{k_i}\dot{k}_i + \tilde{\Lambda}_{k_u}\dot{k}_u + \tilde{\Lambda}_u \dot{u}.$$

Equalizing the right-hand side of this equation and that of (5.4.17), we get

$$\begin{aligned}\tilde{\Lambda}_{k_i}\dot{k}_i + \tilde{\Lambda}_{k_u}\dot{k}_u + \tilde{\Lambda}_u \dot{u} &= \tilde{\Lambda}_0(k_i, k_u, u) \\ &\equiv (r + \mu - \mu\varepsilon)m - \frac{(1+r)(k_i + uk_u) + w_i + uw_u}{(1+u)}.\end{aligned} \tag{5.4.19}$$

From equation (5.4.14), the definitions of $\bar{y}_j$ and $m = \tilde{\Lambda}(k_i, k_u, u)$, we see that π is a function of u, k_i, and k_u. From $(r+\pi)m_j = \varepsilon \bar{y}_j$ and $\bar{y}_j = (1+r)k_j + w_j + \mu m$, we have

$$m_j = \tilde{\Lambda}_j(k_i, k_u, u) \equiv \frac{(1+r)\varepsilon k_j + \varepsilon w_j + \varepsilon\mu m}{r+\pi}, \quad j = i, u. \tag{5.4.20}$$

Taking derivatives of equations (5.4.20) with respect to t yields

$$\dot{m}_j = \frac{\partial \tilde{\Lambda}_j}{\partial k_i}\dot{k}_i + \frac{\partial \tilde{\Lambda}_j}{\partial k_u}\dot{k}_u + \frac{\partial \tilde{\Lambda}_j}{\partial u}\dot{u}, \quad j = i, u. \tag{5.4.21}$$

Substituting $s_j = \lambda \bar{y}_j$ and $a_j = k_j + m_j$ into equations (5.4.6) yields

$$\dot{k}_j = \lambda \bar{y}_j - k_j - \tilde{\Lambda}_j - \dot{m}_j.$$

Insert equations (5.4.21) into the above equations

$$\begin{aligned}\left(1 + \frac{\partial \tilde{\Lambda}_j}{\partial k_i}\right)\dot{k}_i + \frac{\partial \tilde{\Lambda}_j}{\partial k_u}\dot{k}_u + \frac{\partial \tilde{\Lambda}_j}{\partial u}\dot{u} &= \lambda \bar{y}_i - k_i - \tilde{\Lambda}_i, \\ \frac{\partial \tilde{\Lambda}_u}{\partial k_i}\dot{k}_i + \left(1 + \frac{\partial \tilde{\Lambda}_u}{\partial k_u}\right)\dot{k}_u + \frac{\partial \tilde{\Lambda}_u}{\partial u}\dot{u} &= \lambda \bar{y}_u - k_u - \tilde{\Lambda}_u.\end{aligned} \tag{5.4.22}$$

We can write equations (5.3.14) and (5.4.22) in the matrix form

$$\begin{bmatrix} 1+\dfrac{\partial \Lambda_i}{\partial k_i} & \dfrac{\partial \Lambda_i}{\partial k_u} & \dfrac{\partial \Lambda_i}{\partial u} \\ \dfrac{\partial \Lambda_u}{\partial k_i} & 1+\dfrac{\partial \Lambda_u}{\partial k_u} & \dfrac{\partial \Lambda_u}{\partial u} \\ \tilde{\Lambda}_{k_i} & \tilde{\Lambda}_{k_u} & \tilde{\Lambda}_u \end{bmatrix} \begin{bmatrix} \dot{k}_i \\ \dot{k}_u \\ \dot{u} \end{bmatrix} = \begin{bmatrix} \lambda \bar{y}_i - k_i - \tilde{\Lambda}_i \\ \lambda \bar{y}_u - k_u - \tilde{\Lambda}_u \\ \tilde{\Lambda}_0 \end{bmatrix}. \tag{5.4.23}$$

Lemma 5.4.1[15]

The motion of the variables, $k_i(t), k_u(t)$, and $u(t)$ is determined by the three differential equations (5.4.23). For any positive $k_i(t), k_u(t)$, and $u(t)$ at any point of time, all the other variables are uniquely determined by the following procedure: τ by (5.4.11) $\rightarrow N_i$ and N_u by (5.4.12) $\rightarrow K = k_i N_i + k_u N_u \rightarrow F = K^{\alpha}(eN_i)^{\beta}$ $\rightarrow w_u$ by (5.4.13) $\rightarrow r$ by (5.4.16) $\rightarrow m$ by (5.4.18) $\rightarrow m_j, j = i, u$, by (5.4.20) $\rightarrow \bar{y}_j = (1+r)k_j + w_j + \mu m \rightarrow \pi$ by (5.4.14) $\rightarrow c_j$ and s_j by (5.4.5) $\rightarrow a_j = k_j + m_j$.

The lemma is important as it provides a procedure for computing the motion of the system. For any proper initial conditions, it is straightforward to simulate the model with computer. As the expressions are tedious, we do not discuss under what conditions the dynamic system has meaningful solutions. We now study steady states.

5.4.3 The existence of steady states

This section examines steady states. At steady states we have $\pi = \mu$ and $\lambda \bar{y}_j = a_j$. From $\lambda \bar{y}_j = a_j$ and $a_j = k_j + m_j$, we have

$$m_j = (\lambda - 1 + \lambda r)k_j + \lambda w_j + \lambda \mu m, \tag{5.4.24}$$

where we use the definitions of $\bar{y}_j$. From $\pi = \mu$ and $(r+\pi)m_j = \varepsilon \bar{y}_j$, we have $(r+\mu)m_j = \varepsilon \bar{y}_j$. Substituting the definitions of $\bar{y}_j$ into these equations, we obtain

$$(r+\mu)m_j = (1+r)\varepsilon k_j + \varepsilon w_j + \varepsilon \mu m. \tag{5.4.25}$$

Deleting m_j from equations (5.4.24) and (5.4.25), we have

$$[(r+\mu)(\lambda - 1 + \lambda r) - (1+r)\varepsilon]k_j = [\varepsilon - (r+\mu)\lambda](w_j + \mu m), \quad j = i, u. \tag{5.4.26}$$

Dividing the above two equations, we obtain

$$\frac{k_i}{k_u} = \frac{w_i + \mu m}{w_u + \mu m}.$$

From the above equation, $\bar{y}_i = \theta \bar{y}_u$ and $\bar{y}_j = (1+r)k_j + w_j + \mu m$, we have

$$(1+r)(k_u + w_u + \mu m)\left(\frac{k_i}{k_u} - \theta\right) = 0.$$

Hence, we have

$$k_u = \frac{k_i}{\theta}, \quad w_u = \frac{w_i + (1-\theta)\mu m}{\theta}. \tag{5.4.27}$$

From equations (5.4.25) and (5.4.7), we have

$$\begin{aligned} m &= \Lambda(k_i, u) \\ &\equiv \left[(\delta\varepsilon k_i + w_i)\left(1 + \frac{u}{\theta}\right) + \varepsilon w_1\right]\left[\left(\frac{w_1}{(1+u/\theta)k_i} - \delta_1\right)(1+u) - \theta_0 u\right]^{-1}, \end{aligned} \tag{5.4.28}$$

where we use equations (5.4.27), we use

$$r = \frac{w_1}{(1+u/\theta)k_i} - \delta_k, \tag{5.4.29}$$

and

$$\delta \equiv 1 - \delta_k, \quad w_1 \equiv \frac{\alpha w_i}{\beta}, \quad \delta_1 \equiv \delta_k - \mu + \varepsilon\mu, \quad \theta_0 \equiv \frac{(1-\theta)\mu}{\theta}.$$

Substituting $F = AK^{\alpha}N^{*\beta}$ into $w_u N_u = \tau F$ yields

$$w_u u = \tau e^{\beta} A k_i^{\alpha}\left(1 + \frac{u}{\theta}\right)^{\alpha}, \tag{5.4.30}$$

where we use $K = (k_i + uk_u)N_i$. Insert equations (5.4.27) and (5.4.11) in the above equation

$$(1-\theta)\mu\Lambda(k_i, u) = \left[\left(1 + \frac{u}{\theta}\right)^{\alpha}\frac{\theta e^{\beta} A k_i^{\alpha}}{u} - \left(\frac{\theta e^{\beta} w_0^{\alpha} A}{u} + w_i\right)\right]. \tag{5.4.31}$$

Multiplying the equation for m_j in (5.4.24) by N_j and adding the two resulted equations, we have

$$\begin{aligned} &(1-\lambda\mu)(1+u)\Lambda(k_i, u)N \\ &= \left(1 + \frac{u}{\theta}\right)\bar{\lambda}k_i + \lambda w_1 + \lambda w_i + \left[\left(1 + \frac{u}{\theta}\right)^{\alpha} k_i^{\alpha} - w_0^{\alpha}\right]\frac{e^{\beta}\lambda A}{u}, \end{aligned} \tag{5.4.32}$$

where we use $r = w_1/k_i(1+u/\theta) - \delta_k$, and $\bar{\lambda} \equiv \lambda - 1 - \lambda\delta_k$. The two equations, (5.4.31) and (5.4.32), contain two variables, k_i and u. It is difficult to explicitly

solve the equations, even though one may easily do it by simulation. If we solve k_i and u, then all the other variables are given by the following procedure: m by (5.4.28) $\rightarrow N_i$ and N_u by (5.4.12) $\rightarrow r$ by (5.4.29) $\rightarrow k_u$ and w_u by (5.4.27) $\rightarrow m_j$ by (5.4.24) $\rightarrow K = k_i N_i + k_u N_u \rightarrow N^* = e(w_i)N_i \rightarrow F = K^{\alpha} N^{*\beta} \rightarrow a_j = k_j + m_j \rightarrow \bar{y}_j = a_j/\lambda \rightarrow c_j$ and s_j by (5.4.5).

5.5 Other causes of unemployment

Different schools dominated theoretical economics in different periods since the end of the Second World War. In the 1950s and the 1960s, the equilibrium theory and the neo-classical growth theory dominated the literature of theoretical economics. Since then less attention has been paid to the neo-classical growth theory in the literature of theoretical economics, even though the general equilibrium theory has not lost its popularity among applied mathematicians and mathematical economists. The disequilibrium economics had its golden era during the 1970s and the 1980s. This is influenced by the disequilibrium economics which holds that prices and wages may not be equal to their perfectly competitive marginal values.[16] When an economic activity is attempted at the sticky price, it is the adjustment of the quantity that leads to a temporary equilibrium. This approach may explain the existence of unemployment, for instance, by taking account of fixed wage rates due to governments, labor unions and other economic or institutional factors. As mentioned before, the current mainstream of theoretical economics has switched to endogenous knowledge and economic growth. Knowledge (or technology or human capital) is everything for explaining economic growth and international trade patterns. Unemployment disappears in most of the growth models with endogenous knowledge. The assumptions related to full employment of the labor force, which are not accepted in the neo-Keynesian economics, are widely accepted in the contemporary literature of economic growth and development.

Causes and consequences of unemployment in market economies has been a central subject of macroeconomics. Whether unemployment represents a genuine failure of markets to clear is an unsettled question. Varied views have been raised in the literature. For some economists, unemployment is largely illusory or the outcomes of unimportant frictions in the process of matching up workers and jobs; while for others it is the result of non-Walrasian features of the economy. We now mention a few causes, which are proposed to explain the existence of unemployment.

Implicit wage contract model

We now introduce implicit wage contract models to explain why wages can be rigid. The basic idea of this theory is that once a contract in labor market is agreed, it is unusual to have renegotiated before its expiry date and individual workers are committed to a given money wage rate for a specified period of time.[17]

The stickiness of money wage rates impedes market clearing. Wage rates are inflexible partly because setting contracts is costly and partly because both firms and workers would benefit. An individual worker benefits from the inflexibility because of the reduction in uncertainty about his wage rate and employment prospects. The firm benefits from keeping the workers even in recession because they do not need spend hiring or training costs when recovery comes.

Consider a firm that uses only labor as input. Its profit in real term $\pi/\overline{p}$ is

$$\frac{\pi}{\overline{p}} = \frac{p}{\overline{p}} F\left(N, \overline{K}\right) - \frac{w}{\overline{p}} N,$$

where π is the nominal profit, $\overline{p}$ the general price level, p is the price level of the product, w the nominal wage rate, N the number of workers, and $F$the output (as function of N and the fixed capital input $\overline{K}$). The first-order condition of profit maximization says that the firm chooses the level of labor input until the marginal product is equal to the marginal cost (wage rate), i.e.

$$\frac{w}{\overline{p}} = \left(\frac{p}{\overline{p}}\right) F_N\left(N, \overline{K}\right).$$

This equation yields the downward-sloping demand curve for labor. The demand changes according to whether the economy is in boom or recession. The higher demand curve $p_b F_N$ is the labor demand in a boom period, and the lower demand curve $p_r F_N$ is the labor demand in a recession period.

The insider and outsider model

The Walrasian assumption for labor market implies that relationships between firms and workers are "temporary" in the sense that once market conditions are changed, relationships between firms and workers would be adapted immediately. However, in reality, firms do not hire workers afresh each period. Many jobs involve long-term attachments and considerable firm-specific skills and information on the part of workers. The possibility of long-term relationships means that wage may not function smoothly and rapidly so that labor market is cleared at all time. Workers would decide whether to stay at the current job according to their long-term expected income streams rather than only current income.

Search and matching theory

Workers are not identical with respect to conditions for employability and firms demand different types of workers at various times at alternative locations. Heterogeneity among workers and jobs is an essential feature of modern service-dominated and knowledge-based economies. It would be a mistake to treat different workers as a single representative worker or labor markets as a large number of well-connected fast-flowing markets. According to this view, each worker and

each job should be considered as distinct and processes of matching up workers and jobs occur through a complex process of search.

Menu cost theory

The efficiency wage theory produces a real rigidity in that the real wage, rather than the nominal wage is determined independent of labor supply. The new Keynesian approach emphasizes nominal rigidities that occur when a price or wage is fixed in nominal terms and does not readily change in response to changes in supply or demand. Menu cost of changing nominal prices and long-term contracts for both wages and prices that are incompletely indexed are used to explain why a firm's marginal cost does not respond instantly to a decline in demand. Menu cost theory shows that some potentially rather trivial microeconomic factors such as the printing of new price lists or menu cost can lead to large welfare losses. Caplin and Spulber (1987) demonstrate that in the presence of imperfect competition, these small menu costs may have a disproportionate effect on aggregate demand. This theory is limited to the framework of imperfect competition where marginal revenue exceeds marginal cost which is incompatible with perfect competition.

Structural frictions

Structural unemployment occurs when there is a mismatch between the skill or location requirements of job vacancies and the present skills or location of members of the labor force.[18] In reality, an economy is divided into numerous separate labor markets that differ in location, working conditions, and skill requirements. In some labor markets demand for labor may be increasing while in some other markets workers may be fired because of changed market conditions. With numerous separate labor markets, vacancies, and unemployed workers can co-exist.

Technological change and unemployment

Technological progress creates new jobs, but also destroys old ones. The labor-saving character destroys jobs; on the other hand, the productivity-improving feature creates new jobs. As technical progress enriches society by elevating productivity, it stimulates demand for goods and services, and therefore creates new jobs. Innovations destroy old jobs through automation, skill-obsolescence, and the bankruptcies associated with the process of creative destruction. There is no theoretically or empirically convinced answer to the question of whether productivity growth is, in general, positively or negatively correlated with unemployment in the short term. For instance, Pissarides (1990) demonstrated that a technical progress that affected all sectors and uniformly would necessarily raise aggregate employment, because firms would be induced to intensify their recruiting efforts for capitalizing on more rapidly growing productivity. But the requirement that technical progress raises productivity equally rapidly in all jobs

and sectors is never satisfied. As shown by the empirical studies by Davis and Haltiwanger (1992), periods of high unemployment tend to be periods of high job turnover, which are likely associated with productivity growth. The central question of technological change and unemployment is differences in job creation and job destruction in association with the change. Finally, labor unions may also become a significant factor for unemployment.[19]

APPENDIX

A.5.1 Unemployment, inflation, and chaos

This section introduces a worker flow model with a nonlinear outflow rate from unemployment. The model is proposed by Neugart.[20] The unemployment $U(t)$ at period t follows the following identity

$$U(t+1) - U(t) = i(L - U(t)) - o(t)U(t), \tag{A.5.1.1}$$

where $i > 0$ denotes the (exogenous) inflow rate, $L = 1$ the labor force, and $o(t)$ is the outflow rate from unemployment. Here, the outflow rate is defined as the fraction of jobs that come to the market at time t to jobs searchers, i.e. $o(t) = J(t)/\{U(t) + d(1 - U(t))\}$, where the parameter $0 < d < 1$ gives the on-the-job searchers as a constant fraction of employed workers and $J(t)$ denotes job creations. We specify $J(t) = J_s + \gamma(m - \pi(t))$, where J_s is the job creation due to the structural characteristics of the economy and $\gamma(m - \pi(t))$ describes the cyclical component of job creations, where γ is a positive parameter, m the exogenous money growth rate, and $\pi(t)$ the inflation rate at t. Hence, the outflow rate from the unemployment can be expressed

$$o(t) = \frac{J_s + \gamma(m - \pi(t))}{U(t) + d(1 - U(t))}.$$

Substituting the above equation into (A.5.1.1) yields

$$U(t+1) = i - \left[(1-i) + \frac{J_s + \gamma(m - \pi(t))}{U(t) + d(1 - U(t))}\right] U(t) \equiv \Lambda(U(t), \pi(t)). \tag{A.5.1.2}$$

Assume that firms cannot raise prices to the same extent as nominal wages increase at t. This assumption is reflected in the requirement of $\delta > 1$ in the following formation of inflation rate dynamics

$$\pi(t) = \frac{1}{\delta}\left(\pi^e(t) + \frac{w_b(t) - w_p}{w_p}\right), \tag{A.5.1.3}$$

where $\pi(t)$ is the inflation rate at t, $\pi^e(t)$ is expected inflation rate at t, $w_b(t)$ is the bargained wage rate at t, and w_p is the price determined real wage (which equals

$(1-\mu)y$, where $0<\mu<1$ is the fixed mark-up and $y=1$ is the constant marginal productivity).

The bargained real wage $w_b(t)$ is given by $w_b(t)=y-(1-b)U(t)$, where $0<b<1$ is the "reservation wage." The inflation expectation is formed by $\pi^e(t)=a\pi(t-1)+(1-a)\pi^e(t-1)$, where $0\le a\le 1$ is the weighted average rate. Under these assumptions, we may rewrite (A.5.1.3) as

$$\pi(t+1)=\frac{1}{\delta}\left(\pi^e(t+1)+\frac{\mu-(1-b)U(t+1)}{1-\mu}\right). \tag{A.5.1.4}$$

From this equation and $\pi^e(t+1)=a\pi(t)+(1-a)\pi^e(t)$, we have

$$\delta\pi(t+1)=a\pi(t)+(1-a)\pi^e(t)+\frac{\mu-(1-b)U(t+1)}{1-\mu}.$$

Substituting this equation to (7.4.4) for t, we solve

$$\pi(t+1)=\frac{a\mu}{\mu_0\delta}+\left(\frac{a}{\delta}+(1-a)\right)\pi(t)-\frac{ab_0}{\mu_0\delta}\Lambda(U(t),\pi(t))\equiv\Phi(U(t),\pi(t)), \tag{A.5.1.5}$$

where $b_0\equiv 1-b$ and $\mu_0\equiv 1-\mu$. The dynamic system consists of (A.5.1.2) and (A.5.1.5), that is

$$\begin{aligned} U(t+1)&=\Lambda(U(t),\pi(t)),\\ \pi(t+1)&=\Phi(U(t),\pi(t)). \end{aligned} \tag{A.5.1.6}$$

A fixed point is given by $U=\Lambda(U,\pi)$ and $\pi=\Phi(U,\pi)$. It is imposed that at a steady state, $\pi^*=m$, the inflation equaling the money growth rate. It can be shown that the system has a unique steady state

$$(U^*,\pi^*)=\left(\frac{\mu+m\mu_0(1-\delta)}{b_0},m\right),$$

provided that $0\le\mu+m\mu_0(1-\delta)\le b_0$. The Jacobian matrix J is given by

$$J=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix},$$

where

$$a_{11} \equiv \frac{\partial \Lambda(U,\pi)}{\partial U}\Big|_{(U^*,\pi^*)} = 1 - i - \frac{(1-U^*)id}{(U^* + d(1-U^*))U^*},$$

$$a_{12} \equiv \frac{\partial \Lambda(U,\pi)}{\partial \pi}\Big|_{(U^*,\pi^*)} = \frac{\gamma U^*}{U^* + d(1-U^*)},$$

$$a_{21} \equiv \frac{\partial \Phi(U,\pi)}{\partial U}\Big|_{(U^*,\pi^*)} = \frac{b_0}{\delta \mu_0}(1 - a - a_{11}),$$

$$a_{22} \equiv \frac{\partial \Phi(U,\pi)}{\partial U}\Big|_{(U^*,\pi^*)} = \frac{1}{\delta}\left(a + (1-a)\delta - \frac{b_0 a_{12}}{\mu_0}\right).$$

It is straightforward to calculate the two eigenvalues. We now provide some numerical examples carried out by Neugart. As discussed by Neugart, it is reasonable to choose the following parameters

$$a = 0.5, \quad \delta = 2, \quad b = 0.5, \quad \gamma = 0.5. \tag{A.5.1.7}$$

Under (A.5.1.7), if we further specify $d = 0.01, m = 0.03, i \approx 0.13199$, then one eigenvalue is equal to -1; a period-doubling bifurcation occurs. It can be shown that under (A.5.1.7), the eigenvalue with a positive root never crosses 1. Mathematically, the dynamic mechanism is similar to bifurcations observed for the Hénon map. The existence of an attractor with inflation rate over unemployment rate can be demonstrated with computer simulation when the parameters are specified as in (A.5.1.7) and $d = 0.01, m = 0.03, i = 0.18$. The attractor looks like a Phillips curve. Nevertheless, in this model, there is not a stable trade-off between the inflation rate and unemployment rate. The system shifts up and down a negatively sloped "curve" erratically. A pair of inflation-unemployment rates today cannot tell where the economy will be in the long term.

6 Preference change and habit formation

The essence of life is change. In the contemporary literature of economic growth, the standard approach to preference change in the neoclassical growth theory is assume that the rate of time preference in the Ramsey utility is changeable as consumption level varies. We will review this approach in Section 6.1. Another important issue in economic growth theory is how habit interacts with economic growth and money. This chapter deals with issues related to preference change and habit persistence in monetary growth models.

6.1 Uzawa's preference change in the Ramsey approach

By preference change, we mean change in attitudes towards savings in different forms, different types of work, food consumption structure, entertainment, different forms and levels of education in different fields, and different kinds of friendships. Preferences change as time passes and socioeconomic conditions vary. Preference and its dynamics play an important role in determination of economic structure and its dynamics. However, preference change, growth, and economic structure have been rarely examined within a compact framework. There are a few studies in optimal growth theory, which examine dynamic preference and economic growth.

Before introducing money into the OSG model, we show how money and preference are introduced to the standard Ramsey model. The model, proposed by Kam and Missios (2003), considers a monetary growth model in which real balance holdings do not yield utility but appear in the definition of real wealth and the asset accumulation identity. Most of the variables are defined as in Section 3.1. Assume that the discount factor that is applied to future streams of utility is

$$\rho(t) = \int_0^t \theta(a(\omega), \omega) d\omega,$$

where $\theta(a(t), t)$ is the rate of time preference that is assumed to be an increasing function of real wealth a $(= k(t) + m(t))$. The representative agent maximizes the following utility

$$U = \int_0^\infty u(c(t))e^{-\rho(t)}dt.$$

where the parameter, ρ, is the rate of time preference.[1] Real per capita accumulation is given by

$$\dot{a}(t) = f(k(t)) + x(t) - c(t) - \pi(t)m(t),$$

where $x(t)$ is the government transfers. The agent also faces the cash-in-advance constraint

$$m(t) = c(t) + \dot{k}(t).$$

The agent maximizes the utility functional subjects to the above two constraints and the following stock budget constraint

$$\lim_{t\to\infty} k(t)\exp\left(-\int_0^t r(\omega)d\omega\right) \geq 0.$$

As the analysis is standard, we will not represent them here. The analytical results on the effects of an increase in μ in the two models, with and without CIA are given in Table 6.1.1.

The traditional growth theory has two main modeling frameworks – the Solow model and the Ramsey model. The Solow model is evidently not proper to deal with preference changes since it does not have a rational mechanism to determine consumer choice. For instance, the Solow model cannot deal with choice between consumption of goods and leisure time, based on rational choice theory. As pointed out by Zhang (2006a), the Ramsey model is not proper for taking account of change in preference structure (for instance, with regards to housing, leisure time and food). The Ramsey model maximizes $\int_0^\infty U(C)\exp(-\rho t)dt$, subject to the dynamic budget constraint of capital accumulation. In the traditional studies of growth and preference change, by

Table 6.1.1 Effects of increases in μ.

Approach	c	m	k
No CIA	−	+	−
CIA	+	−	+

preference change it means changes of ρ.[2] Since ρ is defined with respect to utility, it is conceptually not easy to introduce reasonable dynamics of ρ. It is difficult to discuss issues related to preference change and economic structure within the Ramsey framework. For instance, when $C(t)$ is a vector consisting leisure time and consumption levels of agricultural and industrial goods, it is difficult to see how changes in ρ can take account of possible changes in the preference structure for $C(t)$. Although the preference change introduced by Uzawa has been widely applied in economics for examining different issues, like the Ramsey utility function, its validity has been doubted by economists. For instance, Blanchard and Fischer observe:

> ... the Uzawa function, with its assumption (that the rate of time preference increases in instantaneous utility) is not particularly attractive as a description of preferences and is not recommended for general use.[3]

Turnovsky also cautions against using Uzawa's preference change:

> ... the requirement that the rate of time discount ... must increase with the level of utility and therefore consumption, is not particularly appealing. It implies that, as agents become richer and increase their consumption levels, their preference for consumption over future consumption increases, whereas intuitively, one would expect the opposite to be more likely.[4]

In the next section, we will introduce preference change into the monetary model in the OSG framework.

6.2 The CIA-OSG model with preference change

This section introduces preference change in the CIA-OSG model in Section 4.1.

6.2.1 The model with preference change

The variables, N, $K(t)$, $F(K(t),N)$, $y(t)=f(k(t))$, $k(t)$, $M(t)$, $P(t)$, $m(t)$, and δ_k are defined as in Section 4.1. We fix $N=1$. Like Section 4.1, the marginal conditions for the production sector are

$$r(t)+\delta_k = f'(k(t)), \quad w(t)=f(k(t))-k(t)f'(k(t)). \tag{6.2.1}$$

We have

$$\dot{M}(t)=\mu M(t), \quad \mu>0,$$

$$g(t)=\frac{\dot{M}(t)}{P(t)}=\frac{\mu M(t)}{P(t)}=\mu m(t). \tag{6.2.2}$$

The liquidity constraint of the household is formed as

$$M(t) \geq \chi P(t) c(t), \tag{6.2.3}$$

where χ is a positive parameter. We require $0 < \chi < 1$. From $M = mP$, we have

$$\pi(t) = \frac{\dot{P}(t)}{P(t)} = \mu - \frac{\dot{m}(t)}{m(t)}.$$

The disposable income is given by

$$\hat{y}(t) = r(t)k(t) + w(t) + a(t) - \pi(t)m(t), \tag{6.2.4}$$

where $a(t) \equiv k(t) + m(t)$. The budget constraint is given by $c(t) + s(t) = \hat{y}(t)$. Insert equation (6.2.4) into the budget constraint

$$(1 + \chi\pi(t))c(t) + s(t) = \bar{y}(t) = y(t) + a(t),$$

where we use $m(t) = \chi c(t)$. We specify the utility function in the Cobb-Douglas form

$$U(t) = c^{\xi(t)}(t)s^{\lambda(t)}(t), \quad \xi(t), \lambda(t) > 0, \quad \xi(t) + \lambda(t) = 1.$$

The optimal solution is

$$c(t) = \frac{\xi \bar{y}(t)}{1 + \chi\pi(t)}, \quad s(t) = \lambda \bar{y}(t). \tag{6.2.5}$$

The wealth accumulation is given by

$$\dot{a}(t) = s(t) - a(t). \tag{6.2.6}$$

We now specify preference change. As $\xi(t) + \lambda(t) = 1$ holds at any point of time, it is sufficient to be concerned with $\lambda(t)$. Following Zhang (2006a: Section 11.2), we assume that the propensity to own wealth is affected by $a(t)$ and $y(t)$ as follows

$$\dot{\lambda}(t) = \kappa\left[G(a(t), y(t)) - \lambda(t)\right], \quad 1 > \lambda(t) \geq 0, \quad \infty > \kappa \geq 0,$$

where $y = rk + w$, κ is a positive adjustment parameter and G is a continuous function of a and y. If $\kappa = 0$, λ is constant. If $\kappa \to \infty$, $G(a,y) = \lambda$ is

held at any point of time. For convenience of analysis, we specify G as follows

$$G = \frac{\theta_0}{1+\theta_1 a^{\theta_a} y^{\theta_y}}, \quad 1 > \theta_0 > 0,\ \theta_1 > 0.$$

The requirement of $\theta_1 > 0$ guarantees $G < \theta_0$. If $\theta_a > (<)0$, the propensity to own wealth tends to be reduced (increased) as wealth a rises. If $\theta_a = 0$, the term G is not affected by a. If $\theta_y > (<)0$, the propensity to own wealth tends to be reduced (increased) as the current income y rises. If $\theta y = 0$, the term G is not affected by y. In this section, we will not specify whether a and b are positive or negative as it is difficult to judge, in general, whether an increase in wealth and income will increase or decrease the propensity to own wealth. It should be noted that the function G may be affected by other factors. For instance, preferences for goods may be affected by prices as people may judge quality by price or because of "snob appeal" effects. The preference change is given by

$$\dot{\lambda} = \kappa\left[\frac{\theta_0}{1+\theta_1 a^{\theta_a} y^{\theta_y}} - \lambda\right]. \tag{6.2.7}$$

6.2.2 The dynamics and steady state

From the definitions of $\bar{y}(t)$ and $a(t)$, we have

$$\bar{y}(t) = f(k(t)) + \delta k(t) + m(t), \tag{6.2.8}$$

where we use $y(t) = f(k(t)) - \delta_k k(t)$ and $\delta \equiv 1 - \delta_k$. Assuming inequality (6.2.3) to be at strict equality, we have

$$m(t) = \chi c(t) = \frac{\chi\xi\bar{y}(t)}{1+\chi\pi(t)}. \tag{6.2.9}$$

Inserting $\pi = \mu - \dot{m}/m$ in equation (6.2.9), we have

$$\dot{m} = (\mu_0 - \xi)m - \xi f(k) - \delta\xi k, \tag{6.2.10}$$

where we use equation (6.2.8) and

$$\mu_0 \equiv \frac{1}{\chi} + \mu.$$

We require $\mu_0 > \xi$, otherwise $m(t)$ will fall monotonically as m, f, and k are positive. As this requirement means

$$1 + \chi\mu > \chi\xi,$$

where ξ is equal to the percentage of the total potential disposable income located to consumption, we see that this inequality simply implies that the demand for

money is not larger than the potential disposable income. This requirement is satisfied if χ is not large.

Substituting $s = \lambda\bar{y}$ and $a = k + m$ into (6.2.6) yields

$$\dot{k} = \lambda f(k) - (\xi + \lambda\delta_k)k - \xi m - \dot{m}. \tag{6.2.11}$$

From the definition of y and equations (6.2.1), we have

$$y = f(k) - \delta_k k.$$

Substituting this equation into equation (6.2.7) yields

$$\dot{\lambda} = \kappa\left[\frac{\theta_0}{1 + \theta_1 a^{\theta_a}(f - \delta_k k)^{\theta_y}} - \lambda\right]. \tag{6.2.12}$$

We have established the model. The dynamics are given by the three differential equations (6.2.10)–(6.2.12). The dynamic system contains three variables, $k(t)$, $m(t)$, and $\lambda(t)$. For any positive k, m, and λ, the other variables are uniquely determined as follows: $\bar{y}(t)$ by (6.2.8) $\rightarrow \pi = \mu - \dot{m}/m \rightarrow a = k + m \rightarrow r$ and w by (6.2.1) $\rightarrow c$ and s by (6.2.5).

For simplicity, we specify $f = Ak^{\alpha}$. By equations (6.2.10)–(6.2.12), at steady state we have

$$\begin{aligned}
&(\mu_0 - \xi)m - \xi f(k) - \delta\xi k = 0,\\
&\lambda f(k) - (\xi + \lambda\delta_k)k - \xi m = 0,\\
&\frac{\theta_0}{1 + \theta_1 a^{\theta_a}(f - \delta_k k)^{\theta_y}} - \lambda = 0.
\end{aligned} \tag{6.2.13}$$

From $a = \lambda\bar{y}$ and equation (6.2.9), we have

$$m = \frac{\chi\xi a}{(1 + \chi\mu)\lambda}, \tag{6.2.14}$$

where we use $\pi = \mu$ at steady state. From $a = k + m$ and equation (6.2.14), we solve

$$m = \frac{\chi\xi k}{\lambda + \lambda\chi\mu - \chi\xi}. \tag{6.2.15}$$

From the first two equations in (6.2.13), we have

$$k = \left(\frac{\lambda\mu_0 - \xi}{(\delta\xi + \delta_k)\mu_0 - \delta_k\xi}\right)^{1/\beta} A^{1/\beta}, \tag{6.2.16}$$

where $\beta \equiv 1-\alpha$. Inserting equations (6.2.14)–(6.2.16) in the last equation in (6.2.13), we have

$$\Omega(\lambda) \equiv \frac{\tilde{\theta}\Lambda^{\theta\alpha/\beta}\lambda^{\theta_a}\xi^{\theta_y}}{(\mu_0\xi+\delta_k\Lambda)^{\theta/\beta}} - \frac{\theta_0}{\lambda} + 1 = 0, \tag{6.2.17}$$

where $\theta \equiv \theta_a + \theta_y$ and

$$\tilde{\theta} \equiv \theta_1 \mu_0^{\theta} A^{\theta/\beta}, \quad \Lambda(\lambda) \equiv (1+\mu_0)\lambda - 1.$$

In order that λ is meaningful, we should require: $1/(1+\mu_0) \le \lambda \le \theta_0$. As

$$\Omega(\theta_0) > 0,$$

$$\Omega\left(\frac{1}{1+\mu_0}\right)\begin{cases} < 0, \text{if } \theta_a+\theta_y > 0. \\ > 0, \text{if } \theta_a+\theta_y < 0, \end{cases}$$

we see that if $\theta_a+\theta_y > 0$, then the problem has at least one solution; if $\theta_a+\theta y < 0$, the problem has either no solution or multiple solutions. We calculate

$$\frac{d\Omega}{d\lambda} = \frac{\theta_0}{\lambda^2} + \left[\frac{(\mu_0-(1+\mu_0)\delta_k\beta)\Lambda+(1+\mu_0)\alpha\mu_0\xi}{(\mu_0\xi+\delta_k\Lambda)\beta\Lambda/\theta} + \frac{\theta_a-\theta\lambda}{\xi\lambda}\right]\frac{\tilde{\theta}\Lambda^{\theta\alpha/\beta}\lambda^{\theta_a}\xi^{\theta_y}}{(\mu_0\xi+\delta_k\Lambda)^{\theta/\beta}}.$$

If $\mu_0 \ge (1+\mu_0)\delta_k\beta$ and $\theta_a \ge \theta\theta_0$, then $d\Omega/d\lambda > 0$. As δ_k is small and $\beta < 1$, it is reasonable to require $\mu_0 \ge (1+\mu_0)\delta_k\beta$. The condition, $\theta_a \ge (\theta_a+\theta_y)\theta_0$, is satisfied if θ_y is negative or relatively small in comparison to θ_a. For instance, if the current income has no impact on the propensity to save, that is, $\theta_y = 0$, then we always have $\theta_a \ge \theta\theta_0$. For explicitly interpreting our analysis, in the reminder of this section we require

$$\theta_a+\theta_y > 0, \quad \mu_0 \ge (1+\mu_0)\delta_k\beta, \quad \theta_a \ge \theta\theta_0.$$

From the above discussions, we see that equation (6.2.17) has a unique positive solution, $1/(1+\mu_0) \le \lambda \le \theta_0$.

Lemma 6.2.1

The dynamics are given by the three differential equations (6.2.10)–(6.2.12). The dynamic system contains three variables, $k(t)$, $m(t)$, and $\lambda(t)$. If we specify $f = Ak^{\alpha}$ and assume

$$\theta_a+\theta_y > 0, \quad \mu_0 \ge (1+\mu_0)\delta_k\beta, \quad \theta_a \ge \theta\theta_0,$$

then the dynamic system has a unique steady state.

Although it is straightforward to calculate the three eigenvalues, we omit stability issues as the expressions are tedious and we cannot explicitly judge the results in general.[5]

6.2.3 Comparative statics analysis

This section examines effects of changes in the inflation policy, the total productivity and the parameter, θ_0.

Effects of the inflation policy

First, we examine effects of changes in the inflation policy on the economic equilibrium. Taking derivatives of equation (6.2.17) with respect to μ yields

$$\frac{d\Omega}{d\lambda}\frac{d\lambda}{d\mu} = -\left[\frac{\theta}{\mu_0} + \frac{(\alpha\mu_0\xi - \beta\delta_k\Lambda)\theta\lambda}{\beta\Lambda(\mu_0\xi + \delta_k\Lambda)}\right]\frac{\tilde{\theta}\Lambda^{\theta\alpha/\beta}\lambda^{\theta_a}\xi^{\theta_y}}{(\mu_0\xi + \delta_k\Lambda)^{\theta/\beta}} < 0,$$

where $d\Omega/d\lambda > 0$. A rise in the inflation rate reduces the propensity to save. From equations (6.2.14)–(6.2.16), we obtain

$$\frac{dk}{d\mu} = \left(\mu_0^2\frac{d\lambda}{d\mu} + \xi^2\right)\frac{A}{\left[(\delta\xi + \delta_k)\mu_0 - \delta_k\xi\right]^2\beta k^{-\alpha}} > 0,$$

$$\frac{1}{m}\frac{dm}{d\mu} = \frac{1}{k}\frac{dk}{d\mu} - \frac{(1+\chi\mu)}{(\lambda + \lambda\chi\mu - \chi\xi)\xi}\frac{d\lambda}{d\mu} - \frac{\lambda\chi}{\lambda + \lambda\chi\mu - \chi\xi},$$

$$\frac{da}{d\mu} = \left(1 + \frac{m}{k}\right)\frac{dk}{d\mu} - \frac{(1+\chi\mu)m}{(\lambda + \lambda\chi\mu - \chi\xi)\xi}\frac{d\lambda}{d\mu} - \frac{\lambda\chi m}{\lambda + \lambda\chi\mu - \chi\xi}.$$

The per-capita physical capital is increased. The real money balances may either increase or decrease. If the money balances rise, the total wealth is increased. From $c = \xi a/\lambda(1+\chi\mu)$, we get

$$\frac{1}{c}\frac{dc}{d\mu} = \frac{1}{a}\frac{da}{d\mu} - \frac{1}{\xi\lambda}\frac{d\lambda}{d\mu} - \frac{\chi}{1+\chi\mu}.$$

As the propensity to save falls and the impact on the total wealth is ambiguous, the impact on the per-capita consumption level is ambiguous. The effects on the per-capita output, the rate of interest and wage rate are given by

$$\frac{df}{d\mu} = A\alpha k^{-\beta}\frac{dk}{d\mu} > 0, \quad \frac{dr}{d\mu} = -A\alpha\beta k^{-\beta-1}\frac{dk}{d\mu} < 0,$$

$$\frac{dw}{d\mu} = A\alpha\beta k^{-\beta}\frac{dk}{d\mu} > 0.$$

The output rises, the rate of interest falls, and the wage rate falls.

Effects of the total productivity

Taking derivatives of equation (6.2.17) with respect to A yields

$$\frac{d\Omega}{d\lambda}\frac{d\lambda}{dA} = -\frac{\theta\tilde{\theta}\Lambda^{\theta\alpha/\beta}\lambda^{\theta_a}\xi^{\theta_y}}{(\mu_0\xi + \delta_k\Lambda)^{\theta/\beta}\beta A} < 0.$$

As the total productivity is increased, the propensity to save falls. From equations (6.2.14)–(6.2.16), we obtain

$$\beta k^{-\alpha}\frac{dk}{dA} = \frac{A\mu_0^2}{\left[(\delta\xi+\delta_k)\mu_0-\delta_k\xi\right]^2}\frac{d\lambda}{dA} + \frac{\lambda\mu_0-\xi}{(\delta\xi+\delta_k)\mu_0-\delta_k\xi},$$

$$\frac{1}{m}\frac{dm}{dA} = \frac{1}{k}\frac{dk}{dA} - \frac{(1+\chi\mu)}{(\lambda+\lambda\chi\mu-\chi\xi)\xi}\frac{d\lambda}{dA}.$$

$$\frac{da}{dA} = \left(1+\frac{m}{k}\right)\frac{dk}{dA} - \frac{(1+\chi\mu)}{(\lambda+\lambda\chi\mu-\chi\xi)\xi}\frac{d\lambda}{dA}.$$

Since the propensity to save falls, the impact on the per-capita physical wealth is ambiguous. From $c=\xi a/\lambda(1+\chi\mu)$, we get

$$\frac{1}{c}\frac{dc}{dA} = \frac{1}{a}\frac{da}{dA} - \frac{1}{\xi}\frac{d\lambda}{dA}.$$

The effects on the per-capita output, the rate of interest and wage rate are given by

$$\frac{df}{dA} = k^{\alpha} + A\alpha k^{-\beta}\frac{dk}{dA},$$

$$\frac{dr}{dA} = \alpha k^{-\beta} - A\alpha\beta k^{-\beta-1}\frac{dk}{dA},$$

$$\frac{dw}{dA} = \beta k^{\alpha} + A\alpha\beta k^{-\beta}\frac{dk}{dA}.$$

Effects of θ_0 on the propensity to save

Taking derivatives of equation (6.2.17) with respect to θ_0 yields

$$\frac{d\Omega}{d\lambda}\frac{d\lambda}{d\theta_0} = \frac{1}{\lambda} > 0.$$

A rise in θ_0 increases the propensity to save. From equations (6.2.14)–(6.2.16), we obtain

$$\beta k^{-\alpha}\frac{dk}{d\theta_0} = \frac{A\mu_0^2}{\left[(\delta\xi+\delta_k)\mu_0-\delta_k\xi\right]^2}\frac{d\lambda}{d\theta_0} > 0,$$

$$\frac{1}{m}\frac{dm}{d\theta_0} = \frac{1}{k}\frac{dk}{d\theta_0} - \frac{(1+\chi\mu)}{(\lambda+\lambda\chi\mu-\chi\xi)\xi}\frac{d\lambda}{d\theta_0},$$

$$\frac{da}{d\theta_0} = \left(1+\frac{m}{k}\right)\frac{dk}{d\theta_0} - \frac{(1+\chi\mu)m}{(\lambda+\lambda\chi\mu-\chi\xi)\xi}\frac{d\lambda}{d\theta_0}.$$

From $c = \xi a/\lambda(1+\chi\mu)$, we get

$$\frac{1}{c}\frac{dc}{d\theta_0} = \frac{1}{a}\frac{da}{d\theta_0} - \frac{1}{\xi\lambda}\frac{d\lambda}{d\theta_0}.$$

The effects on the per-capita output, the rate of interest and wage rate are given by

$$\frac{df}{d\mu} = A\alpha k^{-\beta}\frac{dk}{d\theta_0} > 0,$$

$$\frac{dr}{d\theta_0} = -A\alpha\beta k^{-\beta}\frac{dk}{d\theta_0} < 0,$$

$$\frac{dw}{d\mu} = A\alpha\beta k^{-\beta}\frac{dk}{d\theta_0} > 0.$$

6.3 The MIUF-OSG model with preference change

This section introduces money and preference change into the OSG model with the MIUF approach.

6.3.1 The model with preference change

The variables, $N, K(t), F(K(t), N), y(t) = f(k(t)), k(t), M(t), P(t), m(t)$, and δ_k are defined as in Section 3.3. We fix $N = 1$. Like Section 3.3, the marginal conditions for the production sector are

$$r(t) + \delta_k = f'(k(t)), \quad w(t) = f(k(t)) - k(t)f'(k(t)). \tag{6.3.1}$$

We have

$$\dot{M}(t) = \mu M(t), \quad \mu > 0,$$

$$g(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t),$$

$$\pi(t) = \frac{\dot{P}(t)}{P(t)} = \mu - \frac{\dot{m}(t)}{m(t)}. \tag{6.3.2}$$

The disposable income in real terms, $\hat{y}$, is

$$\hat{y} = rk + w - \pi m + \mu m + a, \tag{6.3.3}$$

where $a(t) \equiv k(t) + m(t)$. We assume that at each point of time the consumer's utility function for holding money, consuming goods and saving is being represented by the following utility function[6]

$$U(t) = m^{\varepsilon(t)}(t)c^{\xi(t)}(t)s^{\lambda(t)}(t), \ \varepsilon(t), \ \xi(t), \ \lambda(t) > 0, \ \varepsilon(t) + \xi(t) + \lambda(t) = 1,$$

where $\varepsilon(t), \xi(t)$, and $\lambda(t)$ are the propensity to hold money, to consume, and to own wealth. The budget constraint can be written as

$$(r+\pi)m+c+s=\bar{y}\equiv f+\delta k+\mu m, \tag{6.3.4}$$

where we use $f=rk+w+\delta_k k$ and $\delta\equiv 1-\delta_k$. Consumers' problem is to choose current money, consumption, and savings in such a way that utility levels are maximized. Maximizing $U(t)$ subject to the budget constraint (6.3.4) yields

$$(r+\pi)m=\varepsilon\bar{y}, \quad c=\xi\bar{y}, \quad s=\lambda\bar{y}. \tag{6.3.5}$$

The real wealth changes as follows

$$\dot{a}(t)=s(t)-a(t). \tag{6.3.6}$$

We now specify preference change. As $\varepsilon(t)+\xi(t)+\lambda(t)=1$ holds at any point of time, we need two equations to determine the preference. For simplicity, we specify

$$\varepsilon(t)=h\xi(t), \tag{6.3.7}$$

where h is a positive constant. This relation means that the propensity to hold money is proportional the propensity to consume goods. If the propensity to consume goods rises, the propensity to hold money rises; and vice versa. From $\varepsilon+\xi+\lambda=1$ and $\varepsilon=h\xi$, we see that if we know the dynamics of $\lambda(t)$, then the preference is determined.

As in Section 6.2, we assume that the propensity to own wealth is affected by $a(t)$ and $y(t)$ as follows

$$\dot{\lambda}(t)=\kappa\left[G(a(t),y(t))-\lambda(t)\right], \quad 1>\lambda(t)\geq 0, \quad \infty>\kappa\geq 0,$$

where $y=rk+w$. For convenience of analysis, we specify G as follows

$$G=\frac{\theta_0}{1+\theta_1 a^{\theta_a} y^{\theta_y}}, \quad 1>\theta_0>0, \quad \theta_1>0.$$

The preference change is given by

$$\dot{\lambda}=\kappa\left[\frac{\theta_0}{1+\theta_1 a^{\theta_a} y^{\theta_y}}-\lambda\right]. \tag{6.3.8}$$

Inserting $\pi=\mu-\dot{m}/m$ in $(r+\pi)m=\varepsilon\bar{y}$ in equation (6.3.5), we have

$$\dot{m}=rm+\mu m-\varepsilon\bar{y}.$$

Substituting the definition of $\bar{y}$ into the above equation yields

$$\dot{m}=\Omega_m(k,m,\lambda)\equiv(r+\mu-\varepsilon\mu)m-(f+\delta k)\varepsilon, \tag{6.3.9}$$

where we use $\varepsilon=(1-\lambda)/(1+1/h)$.

Substituting $s = \lambda\bar{y}$ and $a = k + m$ into (6.3.6) yields

$$\dot{k} = \Omega_k(k, m, \lambda) \equiv \lambda f - (1 - \lambda\delta)k - (1 - \lambda\mu)m - \Omega_m, \tag{6.3.10}$$

where we use the definition of $\bar{y}$.

From the definition of y and equations (6.3.1), we have $y = f(k) - \delta_k k$. Substituting this equation into equation (6.3.8) yields

$$\dot{\lambda} = \Omega_\lambda(k, m, \lambda) \equiv \kappa\left[\frac{\theta_0}{1 + \theta_1(k+m)^{\theta_a}(f - \delta_k k)^{\theta_y}} - \lambda\right]. \tag{6.3.11}$$

We have established the model. The dynamics are given by the three differential equations (6.3.10)–(6.3.11). The dynamic system contains three variables, $k(t)$, $m(t)$, and $\lambda(t)$. It can be seen that for any positive k, m, and λ, all the other variables are uniquely determined.

For simplicity, we specify $f = Ak^\alpha$. By equations (6.3.9)–(6.3.11), at steady state we have

$$\begin{aligned}
&(r + \mu - \varepsilon\mu)m - (f + \delta k)\varepsilon = 0,\\
&\lambda f - (1 - \lambda\delta)k - (1 - \lambda\mu)m = 0,\\
&\frac{\theta_0}{1 + \theta_1(k+m)^{\theta_a}(f - \delta_k k)^{\theta_y}} - \lambda = 0.
\end{aligned} \tag{6.3.12}$$

From $a = \lambda\bar{y}$ and $(r + \pi)m = \varepsilon\bar{y}$, we have

$$m = \frac{(f + \delta k)\varepsilon}{r + \mu - \varepsilon\mu}, \tag{6.3.13}$$

where we use $\pi = \mu$ at steady state and the definition of $\bar{y}$. From the first two equations in (6.3.12), we have

$$k^{2\beta} + f_1(\lambda)k^\beta - f_2(\lambda) = 0, \tag{6.3.14}$$

where we use $f = Ak^\alpha$, $r = \alpha Ak^{-\beta} - \delta_k$, $\beta \equiv 1 - \alpha$, and

$$f_1(\lambda) \equiv \frac{(1 - \lambda\mu)\varepsilon A + (1 - \lambda\delta)\alpha A - (\mu - \varepsilon\mu - \delta_k)A\lambda}{f_0(\lambda)}, \quad f_2(\lambda) \equiv \frac{\alpha A^2\lambda}{f_0(\lambda)},$$

$$f_0(\lambda) \equiv (1 - \lambda\mu)\varepsilon\delta + (\mu - \varepsilon\mu - \delta_k)(1 - \lambda\delta).$$

We note that $1 - \lambda\mu > 0$ and $1 - \lambda\delta > 0$. The sign of f_2 is the same as that of f_0. If $f_2 > 0$, the equation has a unique solution as follows

$$k = \left(\frac{-f_1 + \sqrt{f_1^2 + 4f_2}}{2}\right)^{1/\beta}. \tag{6.3.15}$$

It can be seen that if $\varepsilon > \delta_k$, then it is reasonable to require $f_2 > 0$. If $f_2 < 0$, we have $f_1 < 0$. In this case, the equation may have either none, one or two solutions.

Inserting equations (6.3.13) and (6.3.15) in the last equation in (6.3.13), we have

$$\Lambda(\lambda) \equiv \lambda\theta_1 (k+m)^{\theta_a} (f - \delta_k k)^{\theta_y} + \lambda - \theta_0 = 0. \tag{6.3.16}$$

Equation (6.3.16) involves a single variable. We can examine steady states like in Section 6.2. As we explicitly provided the procedure to determine the motion of the system, it is straightforward to examine the dynamics by simulation.

6.4 A two-sector two-group monetary growth model with consumer durables

This section proposes a two-sector two-group monetary growth model with taste change. The model synthesizes different models proposed in Section 6.3 and in the previous chapters with the MIUF approach.

6.4.1 The two-sector two-group monetary growth model with preference change

Labor is homogeneous and labor grows at an exogenously given exponential rate n which is assumed to be zero. There is only one malleable capital good. Capital goods can be used as an input in both sectors in the economy. Capital depreciates at a constant exponential rate δ_k, which is independent of the manner of use. The two sectors use a single grade of labor and a single type of capital good. The two inputs are smoothly substitutable for each other in each sector and are freely transferable from one sector to the other. Both exogenously determined labor supply and irrevocably existing capital stock are inelastically offered for employment. Both sectors use neoclassical technology with the standard Inada conditions. The production functions are given by $F_j\left(K_j(t), N_j(t)\right), j = i, s$, where F_j are the output of sector j, K_j, and N_j are respectively the capital and labor used in sector j, F_j the production functions, the subscripts i and s denote the capital good sector and the consumer goods sector. Assume F_j to be neoclassical. We have

$$f_j(t) = f_j\left(k_j(t)\right), \quad f_j(t) \equiv \frac{F_j(t)}{N_j(t)}, \quad k_j(t) \equiv \frac{K_j(t)}{N_j(t)}, \quad j = i, s.$$

The functions f_j have the properties as described in Section 2.5. Let N stand for the flow of labor services used at time t for production. We assume that labor is always fully employed. We measure N as follows

$$N = \sum_{j=1}^{2} h_j \overline{N}_j, \tag{6.4.1}$$

where h_j are the levels of human capital of group $j, j = 1, 2$.

We defined $p(t), r(t)$, and $w(t)$ as in Section 2.5. The marginal conditions are given by

$$r+\delta_k = f_i'(k_i) = pf_s'(k_s),$$
$$w(t) = f_i(k_i) - k_i f_i'(k_i) = p(t)\left[f_s(k_s) - k_s f_s'(k_s)\right]. \tag{6.4.2}$$

The wage rate of group j is given by $w_j(t) = h_j w(t)$, $j = 1, 2$.

Total capital stock $K(t)$ is allocated to the two sectors. As full employment of labor and capital is assumed, we have

$$K_i(t) + K_s(t) + \sum_{j=1}^{2} \tilde{K}_j(t) = K(t), \quad N_i(t) + N_s(t) = N. \tag{6.4.3}$$

where $\tilde{K}_j(t)$ is the level of consumer durables of the representative consumer of group $j, j = 1, 2$. We rewrite the above equations as

$$n_i(t)k_i(t) + n_s(t)k_s(t) + \sum_{j}^{2} \tilde{n}_j \tilde{k}_j(t) = k(t),$$
$$n_i(t) + n_s(t) = 1, \tag{6.4.4}$$

where

$$k(t) \equiv \frac{K(t)}{N}, \quad n_q(t) \equiv \frac{N_q(t)}{N}, \quad q = i, s, \quad \tilde{k}_j(t) \equiv \frac{\tilde{K}_j(t)}{\overline{N}_j}, \quad \tilde{n}_j \equiv \frac{\overline{N}_j}{N}, \quad j = 1, 2.$$

Let $\overline{k}_j(t)$ denote per capita level of physical wealth of group j at t. According to the definitions, we have

$$K(t) = \sum_{j=1}^{2} \overline{k}_j(t)\overline{N}_j. \tag{6.4.5}$$

We describe the household behavior similarly to Section 3.4. Household j's disposable income in real terms, $\hat{y}_j(t)$, is

$$\hat{y}_j(t) = r(t)\overline{k}_j(t) + w_j(t) + \mu_j m(t) - \pi(t)m_j(t) + a_j(t), \tag{6.4.6}$$

where $a_j(t) \equiv \overline{k}_j(t) + m_j(t)$. The total money issued equals the total money received by the two groups

$$N_1\mu_1 m(t) + N_2\mu_2 m(t) = N_0 \mu m(t),$$

where $N_0(\equiv N_1+N_2)$ is the population. We require $\mu_j \geq 0$ and

$$n_1\mu_1+n_2\mu_2=\mu,$$

where $n_j \equiv N_j/N_0$. We assume that at each point of time consumers' preferences over real money balance, $m_j(t)$, consumption goods, $c_j(t)$, consumer durables, $\tilde{k}_j(t)$, and savings, $s(t)$, can be represented by the following utility function

$$U_j(t)=m_j^{\varepsilon_{0j}(t)}(t)c_j^{\xi_{0j}(t)}(t)\tilde{k}_j^{\eta_{j0}(t)}(t)s_j^{\lambda_{0j}(t)}(t),\ \varepsilon_{0j}(t),\eta_{j0}(t),\xi_{0j}(t),\lambda_{0j}(t)>0,$$

where ε_{0j} is called household j's propensity to hold money,[7] η_{0j} the propensity to consume, ξ_{0j} propensity to consume, and λ_{0j} propensity to own wealth. The budget constraint is given by

$$(1+r(t))m_j(t)+p(t)c_j(t)+(r(t)+\delta_k)\tilde{k}_j(t)+s_j(t)=\hat{y}_j(t).$$

Insert equation (6.4.6) in the above budget constraints

$$(\pi+r)m_j+pc_j+(r+\delta_k)\tilde{k}_j+s_j=\bar{y}_j\equiv(1+r)\bar{k}_j+w_j+\mu_j m. \tag{6.4.7}$$

Consumers' problem is to choose current money, consumption, and savings in such a way that utility levels are maximized. Maximizing $U_j(t)$ subject to the budget constraint (6.4.5) yields

$$(r+\pi)m_j=\varepsilon_j\bar{y}_j,\ pc_j=\xi_j\bar{y}_j,\ (r+\delta_k)\tilde{k}_j=\eta_j\bar{y}_j\ ,s_j=\lambda_j\bar{y}_j, \tag{6.4.8}$$

where

$$\varepsilon_j\equiv\rho_j\varepsilon_{0j},\quad \xi_j\equiv\rho_j\xi_{0j},\quad \eta_j\equiv\rho_j\eta_{0j},\quad \lambda_j\equiv\rho_j\lambda_{0j},$$
$$\rho_j\equiv\frac{1}{\varepsilon_{0j}+\xi_{0j}+\eta_{0j}+\lambda_{0j}}.$$

The real wealth changes as follows

$$\dot{a}_j(t)=s_j(t)-a_j(t). \tag{6.4.9}$$

The total demand for money is equal to the total supply

$$N_0m(t)=m_1(t)N_1+m_2(t)N_2. \tag{6.4.10}$$

The output of the consumer goods sector is consumed by the households. That is,

$$\sum_{j=1}^{2}c_j(t)\bar{N}_j=F_s(t). \tag{6.4.11}$$

We assume that the preferences changes are influenced by the absolute income and wealth levels, and relative income and wealth levels,[8] and the current preference patterns[9]

$$\dot{\lambda}(t) = H(y(t), a(t), \lambda(t)), \tag{6.4.12}$$

where H is a properly defined function of $y(t), a(t)$, and $\lambda(t)$, in which $y(t)$ is a vector with the current incomes as elements, $a(t)$ is a vector with the wealth as elements, and $y(t)$ is a vector with the propensities of the two groups as elements. We do not specify H in this study as the analysis becomes too complicated. We have thus built the dynamic model.

6.4.2 The dynamics with the Cobb-Douglas production functions for the non-monetary economy

The system is very complicated. For illustration, we are first concerned with the dynamics of the non-monetary economy without preference change. That is, $\varepsilon_0(t) = 0, m_j(t) = m(t) = 0$, and all the propensities are constant. As it is difficult to analyze behavior of the dynamic system with general production functions, for explicit interpretation, we specify production functions in the Cobb-Douglas form:

$$f_j(k_j) = A_j k_j^{\alpha_j}, \quad A_j > 0, \quad 0 < \alpha_j < 1, \quad j = i, s.$$

From equations (6.4.12) and (6.4.2), we directly obtain

$$k_s(t) = \alpha k_i(t),$$

where

$$\alpha \equiv \frac{\beta_i \alpha_s}{\beta_s \alpha_i}, \quad \beta_j \equiv 1 - \alpha_j, \quad j = i, s.$$

The capital intensity of the consumer goods sector is proportional to that of the capital good sector. By $k_s = \alpha k_i$ and $\beta_i f_i = \beta_s p f_s$, we solve

$$p(t) = \frac{\beta_i A_i}{\beta_s \alpha^{\alpha_s} A_s} k_i^{\alpha_i - \alpha_s}(t). \tag{6.4.13}$$

The price of consumer goods is positively related to the technological level of the capital good sector but negatively related to that of the consumer goods sector. The price is positively or negatively related to the capital intensity of the capital good sector, depending on the sign of $\alpha_i - \alpha_s$. If $\alpha_i = \alpha_s$, then the price is constant, $p = A_i / A_s$. In the remainder of this section, we require $\alpha_i > \alpha_s$. If the equality holds,

then labor distribution is invariant in time. The analysis of $\alpha_i \leq \alpha_s$ is similar. The condition $\alpha_i > \alpha_s$ (which implies $\alpha < 1$) guarantees $k_s(t) < k_i(t)$ for any t.

As output of the capital goods sector is equal to the depreciation of capital stock and the net savings, we have

$$\sum_{j=1}^{2} s_j \overline{N}_j - K + \delta_k K = F_i.$$

It is straightforward to show that this equation can be derived from the other equations in the system. According to the definitions, we can rewrite (6.4.11) and the above equation as

$$\begin{aligned} &\sum_{j=1}^{2} \tilde{n}_j c_j = n_s f_s, \\ &\sum_{j=1}^{2} \tilde{n}_j s_j - \delta k = n_i f_i. \end{aligned} \tag{6.4.14}$$

In the non-monetary economy, we have $\overline{y}_j = \hat{y}_j$. Substitute $pc_j = \xi_j \hat{y}_j$ and $s_j = \lambda_j \hat{y}_j$ into (6.4.14)

$$\begin{bmatrix} \xi_1 \tilde{n}_1 & \xi_2 \tilde{n}_2 \\ \lambda_1 \tilde{n}_1 & \lambda_2 \tilde{n}_2 \end{bmatrix} \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} p n_s f_s \\ n_i f_i + \delta k \end{bmatrix}.$$

We solve the above linear equations as

$$\begin{aligned} \hat{y}_1 &= \frac{\beta_i \lambda_2 n_s f_i - (n_i f_i + \delta k)\beta_s \xi_2}{\lambda \beta_s \tilde{n}_1}, \\ \hat{y}_2 &= \frac{\beta_s \xi_1 (n_i f_i + \delta k) - \beta_i \lambda_1 n_s f_i}{\lambda \beta_s \tilde{n}_2}, \end{aligned} \tag{6.4.15}$$

where we use $pf_s = \beta_i f_i / \beta_s$ and $\lambda \equiv \xi_1 \lambda_2 - \xi_2 \lambda_1$ ($\neq 0$ assumed). Substituting (6.4.15) and $r + \delta_k = f_i'(k_i)$ into $(r + \delta_k)\tilde{k}_j = \eta_j \hat{y}_j$, we solve

$$\begin{aligned} \tilde{k}_1 &= \left(\overline{\xi}_2 n_s f_i - n_i f_i - \delta k\right) \xi_2 \tilde{\lambda}_1 k_i^{\beta_i}, \\ \tilde{k}_2 &= \left(n_i f_i + \delta k - \overline{\xi}_1 n_s f_i\right) \xi_1 \tilde{\lambda}_2 k_i^{\beta_i}, \end{aligned} \tag{6.4.16}$$

where

$$\tilde{\lambda}_j \equiv \frac{\eta_j}{\lambda \tilde{n}_j \alpha_i A_i}, \quad \overline{\xi}_j \equiv \frac{\lambda_j \beta_i}{\xi_j \beta_s}, \quad j = 1, 2.$$

Substitute (6.4.16) into (6.4.4)

$$\psi_i(k_i)n_i + \psi_s(k_i)n_s = k, \tag{6.4.17}$$

where

$$\psi_i(k_i) \equiv \frac{1 - A_i\tilde{n}_1\xi_2\tilde{\lambda}_1 + A_i\tilde{n}_2\xi_1\tilde{\lambda}_2}{1 + \delta\tilde{n}_1\xi_2\tilde{\lambda}_1 k_i^{\beta_i} - \delta\tilde{n}_2\xi_1\tilde{\lambda}_2 k_i^{\beta_i}} k_i.$$

$$\psi_s(k_i) \equiv \frac{\alpha + A_i\bar{\xi}_2\tilde{n}_1\xi_2\tilde{\lambda}_1 - A_i\bar{\xi}_1\tilde{n}_2\xi_1\tilde{\lambda}_2}{1 + \delta\tilde{n}_1\xi_2\tilde{\lambda}_1 k_i^{\beta_i} - \delta\tilde{n}_2\xi_1\tilde{\lambda}_2 k_i^{\beta_i}} k_i.$$

Solve (6.4.17) and $n_i + n_s = 1$

$$n_i = \frac{\psi_s(k_i) - k}{\psi_s(k_i) - \psi_i(k_i)}, \quad n_s = \frac{k - \psi_i(k_i)}{\psi_s(k_i) - \psi_i(k_i)}. \tag{6.4.18}$$

The labor distribution between the two sectors is uniquely determined by k and k_i.

Insert $s_j = \lambda_j\hat{y}_j$ into the second equation in (6.4.14)

$$\sum_{j=1}^{2}\tilde{n}_j\lambda_j\hat{y}_j - \delta k = A_i k_i^{\alpha_i} n_i. \tag{6.4.19}$$

Substituting (6.4.6) into (6.4.19) yields

$$\sum_{j=1}^{2}\tilde{n}_j\lambda_j(1+r)\bar{k}_j + \sum_{j=1}^{2}\tilde{n}_j\lambda_j w_j - \delta k = A_i k_i^{\alpha_i} n_i.$$

Insert $r = \alpha_i A_i k_i^{-\beta_i} - \delta_k$ and $w_J = h_j\beta_i A_i k_i^{\alpha_i}$ into the above equation

$$\sum_{j=1}^{2}\tilde{n}_j\lambda_j\left(\alpha_i A_i k_i^{-\beta_i} + \delta\right)\bar{k}_j + \beta_i A_i k_i^{\alpha_i}\sum_{j=1}^{2}\tilde{n}_j\lambda_j h_j - \delta k = A_i k_i^{\alpha_i} n_i.$$

Insert n_i in (6.4.18) into the above equation

$$\sum_{j=1}^{2}\tilde{n}_j\lambda_j\left(\alpha_i A_i k_i^{-\beta_i} + \delta\right)\bar{k}_j + \left(\beta_i A_i\sum_{j=1}^{2}\tilde{n}_j\lambda_j h_j - \frac{A_i\psi_s(t)}{\psi_s(t) - \psi_i(t)}\right)k_i^{\alpha_i}$$

$$= \left(\delta - \frac{A_i k_i^{\alpha_i}(t)}{\psi_s(t) - \psi_i(t)}\right)k. \tag{6.4.20}$$

According to the definitions, we can rewrite (6.4.5) as

$$k(t) = \sum_{j=1}^{2} \tilde{n}_j \bar{k}_j(t). \tag{6.4.21}$$

Substitute (6.4.21) into (6.4.20)

$$\bar{k}_1 = \Lambda\left(k_i, \bar{k}_2\right) \equiv \frac{\psi(k_i) - \psi_2(k_i)\bar{k}_2}{\psi_1(k_i)}, \tag{6.4.22}$$

where

$$\psi(k_i) \equiv \left(\frac{A_i \psi_s(k_i)}{\psi_s(k_i) - \psi_i(k_i)} - \beta_i A_i \sum_{j=1}^{2} \tilde{n}_j \lambda_j h_j\right) k_i^{\alpha_i},$$

$$\psi_1(k_i) \equiv \left(\frac{A_i k_i^{\alpha_i}(k_i)}{\psi_s(k_i) - \psi_i(k_i)} + \lambda_j \alpha_i A_i k_i^{-\beta_i} + \lambda_j \delta - \delta\right) \tilde{n}_1,$$

$$\psi_2(k_i) \equiv \left(\frac{A_i k_i^{\alpha_i}(k_i)}{\psi_s(k_i) - \psi_i(k_i)} + \lambda_j \alpha_i A_i k_i^{-\beta_i} + \lambda_j \delta - \delta\right) \tilde{n}_2.$$

We see that $\bar{k}_1(t)$ is a function of $k_i(t)$ and $\bar{k}_2(t)$. By (6.4.21) and (6.4.22), we solve $k(t)$ as a function of $k_i(t)$ and $\bar{k}_2(t)$

$$k = \Lambda_0\left(k_i, \bar{k}_2\right) \equiv \tilde{n}_1 \Lambda\left(k_i, \bar{k}_2\right) + \tilde{n}_2 \bar{k}_2.$$

By (6.4.18), we see that the labor distribution $n_i(t)$ and $n_s(t)$ are also uniquely determined as functions of $k_i(t)$ and $\bar{k}_2(t)$. According to (6.4.16), we can explicitly express $\hat{y}_j(t)$ as functions of $k_i(t)$ and $\bar{k}_2(t)$ as

$$\hat{y}_j = \Lambda_j\left(k_i, \bar{k}_2\right).$$

We omit giving explicit expressions as it is straightforward and the expressions are tedious.

Insert $s_j = \lambda_j \Lambda_j\left(k_i, \bar{k}_2\right)$ into (6.4.9)

$$\dot{\bar{k}}_j = \lambda_j \Lambda_j\left(k_i, \bar{k}_2\right) - \bar{k}_j, \quad j = 1, 2. \tag{6.4.23}$$

Taking derivatives of (6.4.22) with respect to t yields

$$\dot{\bar{k}}_1 = \left(\frac{\psi' - \bar{k}_2 \psi_2' - \Lambda \psi_1'}{\psi_1}\right) \dot{k}_i - \frac{\psi_2}{\psi_1} \dot{\bar{k}}_2. \tag{6.4.24}$$

Equations (6.4.22)–(6.4.4) contain three variables, $\bar{k}_1, \bar{k}_2$, and k_i. Nevertheless, the original dynamic system is 2-dimensional. To express the dynamics in a

2-dimensional differential equations system, we substitute (6.4.24) and (6.4.22) into (6.4.23) to eliminate $\overline{k}_1(t)$

$$\dot{k}_i = \frac{\lambda_1\psi_1(k_i)\Lambda_1\left(k_i,\overline{k}_2\right) - \psi_1(k_i)\Lambda\left(k_i,\overline{k}_2\right) + \left(\lambda_2\Lambda_2\left(k_i,\overline{k}_2\right) - \overline{k}_2\right)\psi_2(k_i)}{\psi'(k_i) - \overline{k}_2\psi_2'(k_i) - \Lambda\psi_1'(k_i)},$$

$$\dot{\overline{k}}_2 = \lambda_2\Lambda_2\left(k_i,\overline{k}_2\right) - \overline{k}_2 \tag{6.4.25}$$

The 2-dimensional differential equations (6.4.25) contain two variables k_i and $\overline{k}_2$.

Lemma 6.4.1

The dynamics of the economic system is governed by the 2-dimensional differential equations (6.4.25) with $k_i(t)$ and $\overline{k}_2(t)$ as the variables. Moreover, all the other variables can be determined as functions of $k_i(t)$ and $\overline{k}_2(t)$ at any point of time by the following procedure: $\overline{k}_1(t)$ by (6.4.22) $\rightarrow k_s(t) = \alpha k_i(t) \rightarrow k(t)$ by (6.4.21) $\rightarrow p(t)$ by (6.4.13) $\rightarrow n_i(t)$ and $n_s(t)$ by (6.4.18) $\rightarrow r(t)$ and $w(t)$ by (6.4.2) $\rightarrow w_j(t) = h_j w(t), j = 1, 2 \rightarrow \hat{y}_j(t)$ by (6.4.6) $\rightarrow c_j(t)$ and $s_j(t)$ by (6.4.8) $\rightarrow N_i(t) = n_i(t)N$ and $N_s(t) = n_s(t)N_s \rightarrow K(t) = k(t)N \rightarrow K_i(t) = k_i(t)N_i(t)$ and $K_s(t) = k_s(t)N_s(t) \rightarrow F_i(K_i(t), N_i(t))$ and $F_s(K_s(t), N_s(t))$.

We now simulate the model to illustrate dynamic behavior of the non-monetary system.

6.4.3 Simulating the non-monetary economy

We now simulate the model of the previous section. In this section, the depreciation rate is fixed at $\delta_k = 0.07$. We specify the production functions, the population structure, and preferences as follows

$$\begin{pmatrix} A_i \\ A_s \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.9 \end{pmatrix}, \begin{pmatrix} \alpha_i \\ \alpha_s \end{pmatrix} = \begin{pmatrix} 0.34 \\ 0.32 \end{pmatrix}, \begin{pmatrix} \overline{N}_1 \\ \overline{N}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix},$$

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0.87 \\ 0.75 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0.05 \\ 0.12 \end{pmatrix}. \tag{6.4.26}$$

The capital good sector's technological parameter A_i is higher than that of the consumer goods sector. Group 1's level of human capital and propensity to save are both higher than group 2's. The population share of group 1 is only 1/6 percent. Before demonstrating motion of the system over time, we show that the system has an (unique) equilibrium point. At equilibrium, according to (6.4.25), the following equations hold

$$\lambda_1\psi_1(k_i)\Lambda_1\left(k_i,\overline{k}_2\right) - \psi_1(k_i)\Lambda\left(k_i,\overline{k}_2\right) + \left(\lambda_2\Lambda_2\left(k_i,\overline{k}_2\right) - \overline{k}_2\right)\psi_2(k_i) = 0,$$

$$\lambda_2\Lambda_2\left(k_i,\overline{k}_2\right) - \overline{k}_2 = 0.$$

The simulation demonstrates the existence of a unique equilibrium point for the given parameters. The equilibrium values of the variables are as follows:

$$r = 0.072,\ p = 0.789,\ k = 5.417,\ K = 93.431,\ Y = F_i + pF_s = 17.001,$$

$$\begin{pmatrix} n_i \\ n_s \end{pmatrix} = \begin{pmatrix} 0.377 \\ 0.623 \end{pmatrix}, \begin{pmatrix} k_i \\ k_s \end{pmatrix} = \begin{pmatrix} 2.177 \\ 1.988 \end{pmatrix}, \begin{pmatrix} N_i \\ N_s \end{pmatrix} = \begin{pmatrix} 7.172 \\ 11.828 \end{pmatrix},$$

$$\begin{pmatrix} K_i \\ K_s \end{pmatrix} = \begin{pmatrix} 148.30 \\ 848.37 \end{pmatrix}, \begin{pmatrix} F_i \\ F_s \end{pmatrix} = \begin{pmatrix} 6.640 \\ 13.264 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 5.417 \\ 1.204 \end{pmatrix},$$

$$\begin{pmatrix} \bar{k}_1 \\ \bar{k}_2 \end{pmatrix} = \begin{pmatrix} 70.362 \\ 4.614 \end{pmatrix}, \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} = \begin{pmatrix} 80.873 \\ 6.152 \end{pmatrix}, \begin{pmatrix} \tilde{k}_1 \\ \tilde{k}_2 \end{pmatrix} = \begin{pmatrix} 28.389 \\ 5.183 \end{pmatrix},$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 8.198 \\ 1.013 \end{pmatrix}.$$

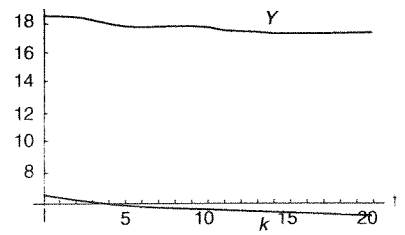
We see that most of the labor force is employed by the consumer goods sector at the equilibrium point. As $\alpha < 1$, the capital intensity of the capital good sector is higher than that of the consumer goods sector. The GDP share of the capital good sector is much lower than that of the consumer goods sector. Group 1's wage rate is higher than that of group 2 as assumed difference in the levels of human capital. Group 1's per capita wealth is much higher than that of group 2; and group 1's consumption levels of consumer goods and consumer durables are higher than group 2's in terms of per capita. The relative economic position of group 1 in the national economy is illustrated as follows:

$$\frac{\overline{N}_1}{\sum_j \overline{N}_j} = 16.7\% \quad \frac{\bar{k}_1 \overline{N}_1}{K} = 75.3\% \quad \frac{y_1 \overline{N}_1}{\sum_j c_j \overline{N}_j} = 87.2\% \quad \frac{\tilde{k}_1 \overline{N}_1}{\sum_j \tilde{k}_j \overline{N}_j} = 52.3\%,$$

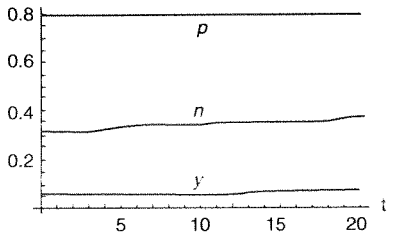
$$\frac{c_1 \overline{N}_1}{\sum_j c_j \overline{N}_j} = 61.8\%.$$

Group 1's population share is only 16.7 percent; but its shares of national wealth and current income are respectively 75.3 percent and 87.2 percent; its shares in consumption of national consumer durables and consumer goods are respectively 52.3 percent and 61.8 percent. The inequality results from differences in levels of human capital and preferences. Evidently, it is important to examine effects of changes in these parameters. Before carrying out comparative statics analysis by simulation, we illustrate the motion of the economy with the initial conditions, $k_i(0) = 2.75$ and $\bar{k}_2(0) = 26$ as in Figure 6.4.1.

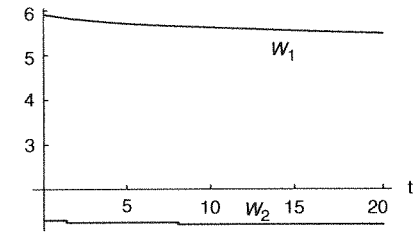
All the variables approach their equilibrium values in the long term. We see that the current national income and the capital stock of per labor input decline over time. The share of the labor force of the capital good sector increases and the rate of interest also slightly increases. The price remains almost constant.



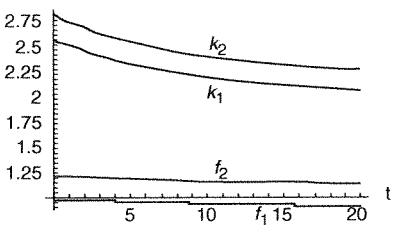
a) The motion of $Y(t)$ and $k(t)$



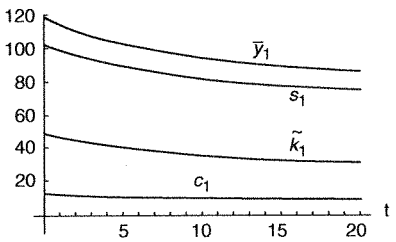
b) The motion of $n_1(t)$, $p(t)$, and $y(t)$



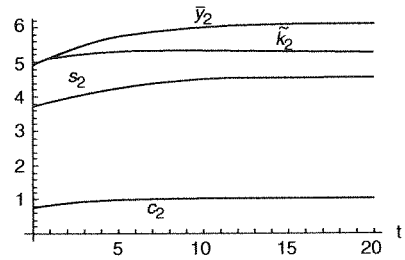
c) $w_1(t)$ and $w_2(t)$



d) $k_1(t)$, $k_2(t)$, $f_1(t)$, and $f_2(t)$



e) $\bar{y}_1(t)$, $s_1(t)$, $\tilde{k}_1(t)$ and $c_1(t)$



f) $\bar{y}_2(t)$, $s_2(t)$, $\tilde{k}_2(t)$ and $c_2(t)$

Figure 6.4.1 The motion with the initial conditions $k_1(0)$ and $\bar{k}_2(0)$.

Group 1's wage rate declines and group 2's wage rate slightly changes. The capita intensities of the two sectors decline and the levels of outputs in terms of per labor inputs decline slightly. The living conditions of group 1 measured in terms of consumption levels of consumer goods and consumer durables decline; the living conditions of group 2 measured in terms of consumption levels of consumer goods and consumer durables are improved.

We now examine impact of changes on dynamic processes of the system. First, we examine the case that all the parameters, except the technology parameter, A_i, are the same as in (6.4.26). We increase the technology level from 0.7 to 0.82. The simulation results are demonstrated in Figure 6.4.2. The solid lines in Figure 6.4.2 are the same as in Figure 6.4.1, representing the values of the corresponding variables when $A_i = 0.7$; the dashed lines in Figure 6.4.2 represent

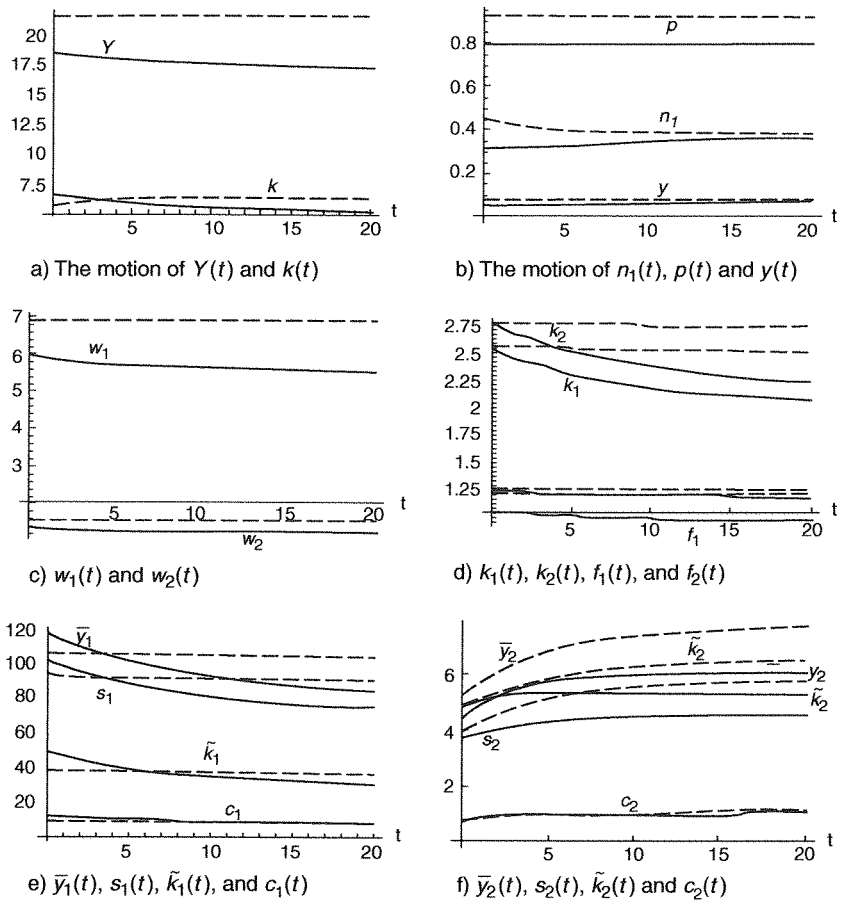


Figure 6.4.2 A_i Rises from 0.7 (solid lines) to 0.81 (dashed lines).

the new values of the variables when $A_i = 0.82$. As the technology level of the capital good sector is improved, the national income increases and $k(t)$ declines initially below the level of the corresponding variable and rises above it after a few periods of time. The price is increased. The share of the labor force in the capital good sector increases initially. The rate of interest is slightly changed due to change in A_i. The wage rates are increased. The capital intensities and the production levels of the two sectors are increased. From Figure 6.4.2e we see that the living conditions of group 2 initially deteriorate and it is only after a few years that the living conditions become better after the technological change. Figure 6.4.2f shows that group 2 benefits during the whole study period.

We now allow the level of group 1's human capital to be changed and remain the rest of the variables constant as in (6.4.26). We increase the level of group 1's

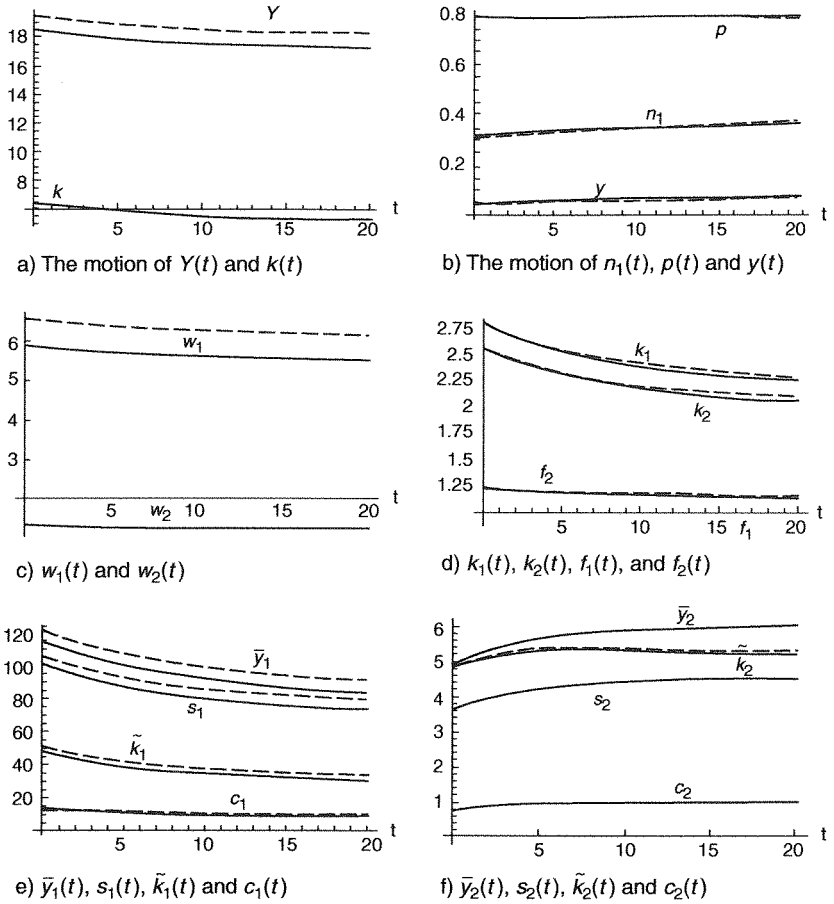


Figure 6.4.3 h_1 Rises from 9 (solid lines) to 10 (dashed lines).

human capital from 9 to 10. The simulation results are demonstrated in Figure 6.4.3. Similarly as in Figure 6.4.2, the solid lines are the same as in Figure 6.4.1 and the dashed lines represent the new values of the variables. We see that the price, the labor force distribution, the capital intensities of the two sector, and the output levels of per labor input, and group 2's living conditions are almost not changed, even though the national income is increased and group 2's living conditions are improved.

Figure 6.4.4 provides the impact of change in group 2's human capital. We increase the level of group 1's human capital from 2 to 3.5. It is important to observe that in the case of group 2's human capital improvement, not only group 2 benefits; but also group 1 benefits greatly. This is quite different from the case when group 1's level of human capital is increased. The improvement of

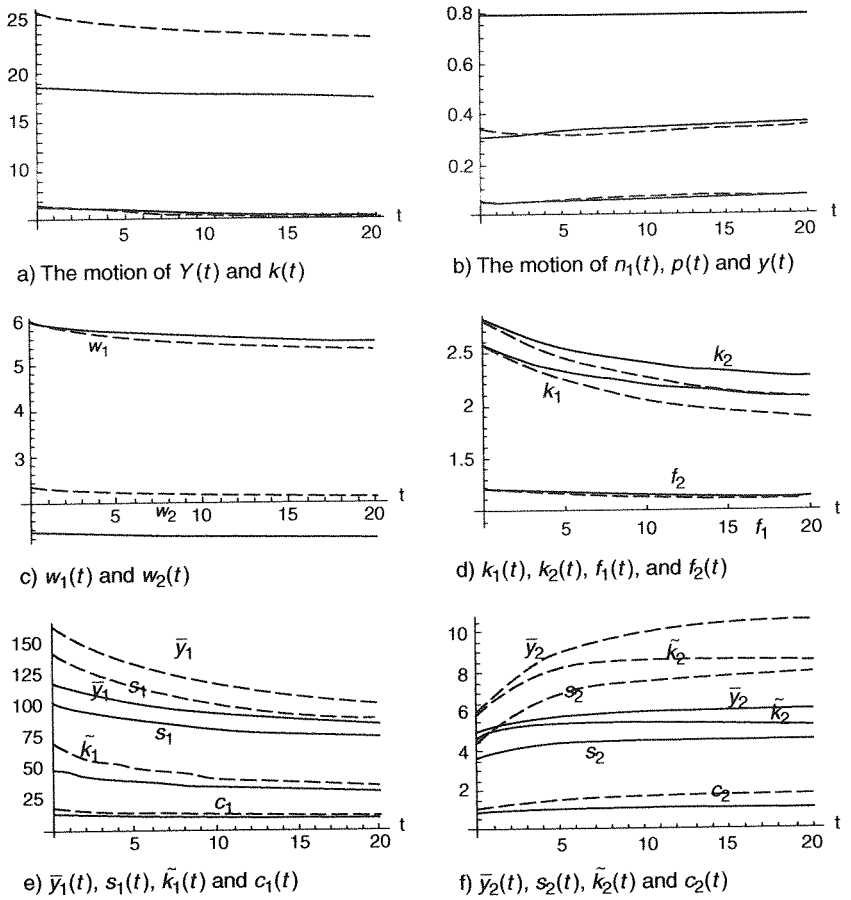


Figure 6.4.4 h_2 Rises from 2 (solid lines) to 3.5 (dashed lines).

human capital of the "rich class" only benefits the rich class; but the improvement of human capital of the poor class benefits all in the economy. Indeed, how this observation can be made is difficult to say as our utility and production functions are specified and the parameter values might not be proper.

Figure 6.4.5 examines the impact of change in group 1's population. We increase group 1's population from 1 to 1.5. It can be see that the national income increases as a consequence of the increase in the population. The price and the rate of interest are slightly affected by the change in the population. The share of the labor force of the capital good sector increases. The wage rates of the two groups are slightly increased. The capital intensities of the two sectors and the output levels of the two sectors in terms of per labor input are increased.

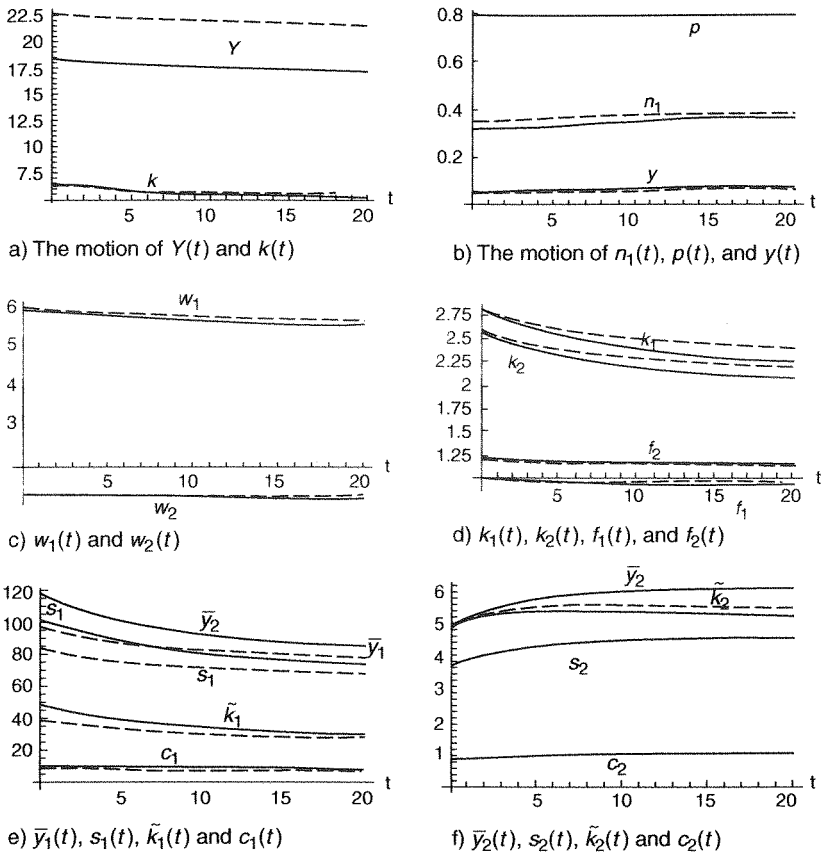


Figure 6.4.5 N_1 Rises from 1 (solid lines) to 1.5 (dashed lines).

Figure 6.4.6 summarizes the impact of change in group 2's population. We increase group 2's population from 5 to 6. It can be seen that the effects are similar to those caused by the change in group 1's population with regard to Y, r, p, w_1, w_2, and k. But the capital intensities of the two sectors and the output levels of the two sectors in terms of per labor input are increased as N_1 rises, while these variables fall as N_2 rises.

In Figure 6.4.7, we give the impact of change in group 1's preference. We specify the preference change as follows: let λ_1 increase from 0.87 to 0.89 and both η_1 and ξ_1 be reduced by 0.01. We see that group 1 does not benefit the preference change in terms of consumption levels of consumer durables and consumer goods, but group 2 benefits.

In Figure 6.4.8, we give the impact of change in group 2's preference. We specify the preference change as follows: let λ_1 increase from 0.75 to 0.79 and both η_2 and ξ_2 be reduced by 0.02. In the contrast to the case that group 1 increases the

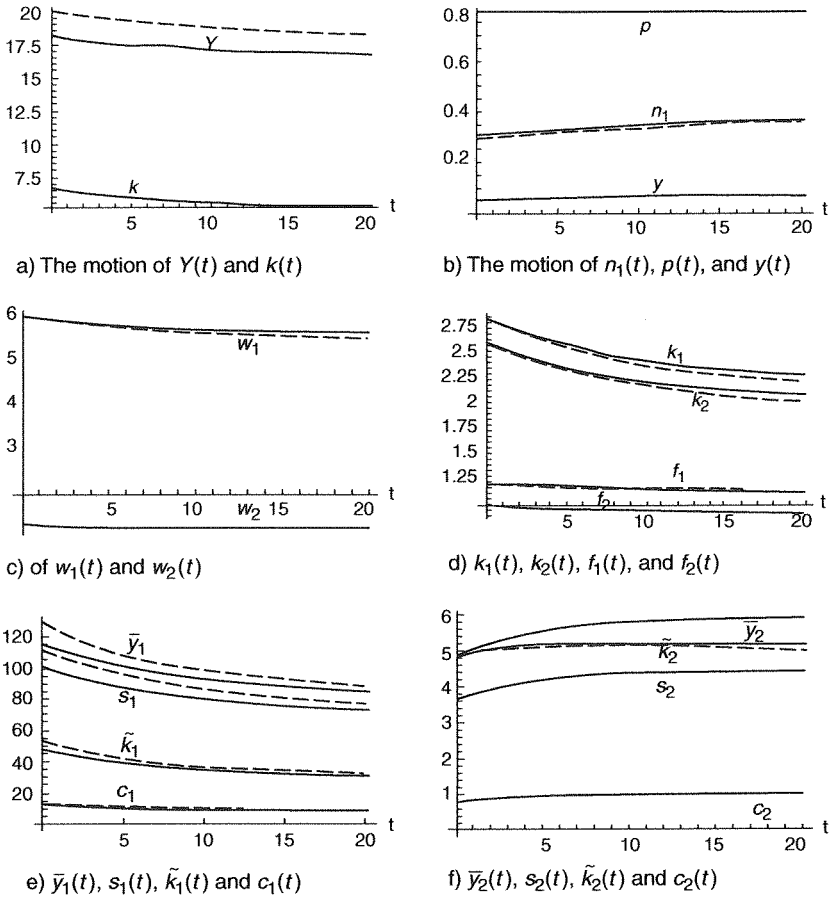
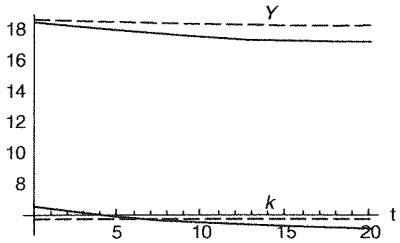


Figure 6.4.6 N_2 Rises from 5 (solid lines) to 6 (dashed lines).

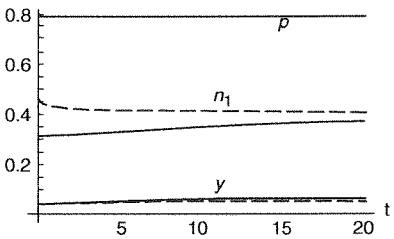
propensity to save, when group 2 increases the propensity to save, the group which changes the preference benefits and the other group is "harmed."

6.5 Habit formation in the Ramsey approach

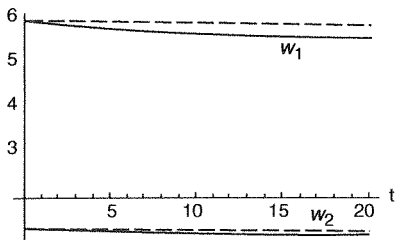
This section is concerned with implications of the idea that preferences exhibit persistence (Ryder and Heal, 1973) for economic growth. There has been growing interest on the effects of habit formation in consumption in the recent macroeconomic literature.[10] For illustrating the basic ideas in this approach, we introduce a MIUF growth model with habit formation by Faria (2001). Faria introduces the hypothesis of habit formation in consumption into the Sidrauski model. As most aspects of the model are similar to the Sidrauski model in Section 3.1, we only illustrate how the model is built.



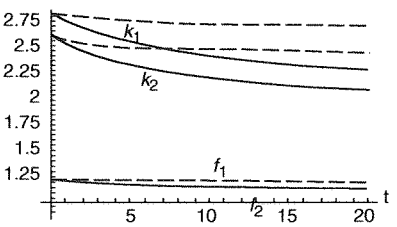
a) The motion of $Y(t)$ and $k(t)$



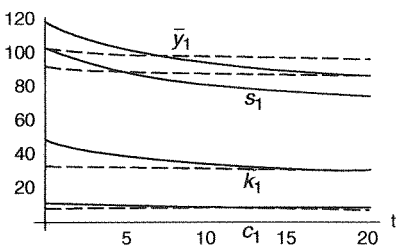
b) The motion of $n_1(t)$, $p(t)$, and $y(t)$



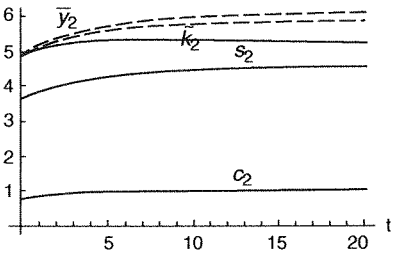
c) $w_1(t)$ and $w_2(t)$



d) $k_1(t)$, $k_2(t)$, $f_1(t)$, and $f_2(t)$



e) $\bar{y}_1(t)$, $s_1(t)$, $\tilde{k}_1(t)$ and $c_1(t)$



f) $\bar{y}_2(t)$, $s_2(t)$, $\tilde{k}_2(t)$ and $c_2(t)$

Figure 6.4.7 λ_1 Rises from 0.87 (solid lines) to 0.89 (dashed lines).

The individual maximizes the following utility functional

$$U = \int_0^\infty \frac{[(c/h^\gamma)^a m^{1-a}]^{1-\sigma}}{1-\sigma} e^{-\rho t} dt,$$

where $c(t)$, $m(t)$, and ρ are defined as in Section 3.1, $h(t)$ is the stock of habits, a $(0 < a < 1)$ is a parameter, γ $(0 < \gamma < 1)$ indexes the importance of habits. If $\gamma = 0$, habit stock has no relevance and the model is equal to the Sidrauski model. If $\gamma = 1$, we see that consumption relative to habit stock is very important. It is assumed that the stock of habits evolves according to

$$\dot{h}(t) = \theta(c(t) - h(t)),$$

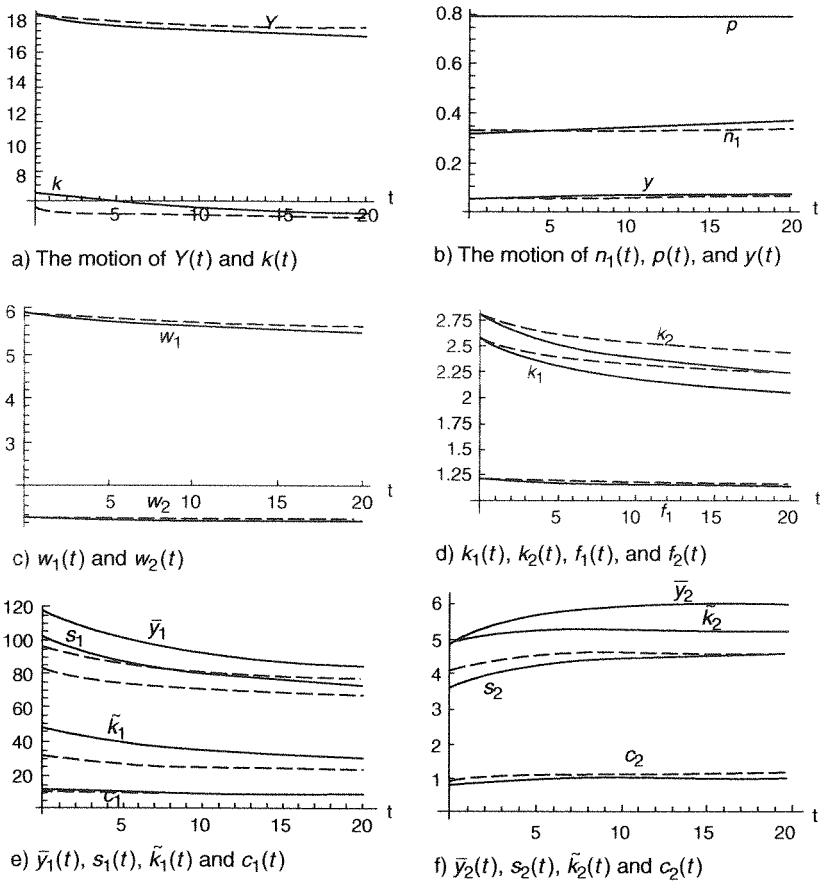


Figure 6.4.8 λ_2 Rises from 0.75 (solid lines) to 0.79 (dashed lines).

where θ is a non-negative parameter determining the relative weights of consumption at different times. This equation can be derived by assuming that habits are a weighted sum of past level of current full consumption, measured by a function, $\omega(c, m)$, of c and m, with exponentially declining weights given to more distant values of ω, as follows

$$h(t) = \theta e^{-\theta t} \int_{-\infty}^{t} e^{\theta \tau} \omega(c(\tau), m(\tau)) d\tau.$$

A larger value for θ would involve lower weights given to more distant values of ω. The evolution of the variable is given b the above differential equation (with $\omega = c$).[11]

Real per capita accumulation is given by

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) + x(t) - c(t) - (r(t) + \pi(t))m(t),$$

where

$$a(t) \equiv k(k) + m(t),$$

and $x(t)(= \mu m(t) = (\pi + n)m(t))$ is the government transfers, $r(t)$ and $w(t)$ are the rate of interest and wage (where $r = f'(k)$, $w = f(k) - kf'(k)$). The problem is subject to the usual no-Ponzi game condition

$$\lim_{t \to \infty} a(t) \exp - \left[\int_0^t (r(s) - n) ds \right] = 0.$$

We will not analyze this model and the interested reader is referred to Faria's original paper.[12]

APPENDIX

A.6.1 The two-sector multi-group non-monetary model with general production functions

We now analyze dynamics when the production functions in the non-monetary economy take on general forms. First, we show that the motion of the system is determined by 2-dimensional differential equations. From $r(t) + \delta_k = f_i'(k_i) = pf_s'(k_s)$ in (6.4.1), we get

$$p = \frac{f_i'(k_i)}{f_s'(k_s)}.$$

Substituting this equation into $f_i(k_i) - k_i f_i'(k_i) = p\left(f_s(k_s) - k_s f_s'(k_s)\right)$ in (6.4.1) yields

$$\Psi_i(k_i) \equiv \frac{f_i(k_i)}{f_i'(k_i)} - k_i = \frac{f_s(k_s)}{f_s'(k_s)} - k_s \equiv \Psi_s(k_s). \tag{A.6.4.1}$$

From the properties of $f_i(k_i)$, we can show that the function $\Psi_i(k_i)$ has the following properties:

$$\Psi_i(0) = 0, \ \Psi_i(k_i) > 0 \text{ for } k_i > 0, \quad \Psi_i'(k_i) = -\frac{f_i(k_i) f_i''(k_i)}{f_i'^2(k_i)} \geq 0.$$

The function $\Psi_s(k_s)$ has the same properties in k_s. We see that for any given $k_s \geq 0$, equation (A.6.4.1) determines $k_s \geq 0$ as a unique function of k_i, denoted by

$k_s = \Omega(k_i)$. We have $\Omega(0) = 0$ and $\Omega' > 0$. From equations (6.4.3), we solve the labor distribution as functions of k, k_s, and k_i as follows

$$n_i = \frac{k - k_s}{k_i - k_s}, \quad n_s = \frac{k_i - k}{k_i - k_s}. \tag{A.6.4.2}$$

The labor distribution is uniquely determined by k and k_s. We should require that $k_i(t) - k_s(t)$ does not switch the sign of $k_i(0) - k_s(0)$ for any t; otherwise, for some $t > 0, 0 < n_i(t), n_s(t) < 1$ would not hold. We assume $k_i(t) > k_s(t)$ for any positive t. For $0 < n_i(t), n_s(t) < 1$ under Assumption A1, we should require $k_i(t) > k(t) > \Omega(k_i(t))$. Insert $p(t)c_j = \xi_j \hat{y}_j$ and equations (6.4.4) into equation (6.4.8)

$$(1+r)\sum_{j=1}^{2} \overline{k}_j \xi_j \overline{N}_j + w\sum_{j=1}^{2} h_j \xi_j \overline{N}_j = pF_s,$$

where we use $w_j = h_j w$. Substitute r and w in (6.4.2) into the above equation

$$\overline{\Omega}_0(k_i)\sum_{j=1}^{2} \overline{n}_j \overline{k}_j + \overline{\Omega}(k_i) - (k_i - k) f_s^*(k_i) = 0, \tag{A.6.4.3}$$

where we use $F_s = f_s n_s N$ and (A.6.4.2) and

$$f_s^*(k_i) \equiv \frac{f_s(\Omega) f_i'(k_i)}{f_s'(\Omega)}, \quad \overline{\Omega}_0(k_i) \equiv (k_i - \Omega(k_i))\left(f'(k_i) + \delta\right),$$

$$\overline{\Omega}(k_i) \equiv (k_i - \Omega(k_i))\left(f(k_i) - k_i f'(k_i)\right)\frac{\sum_j h_j \xi_j \overline{N}_j}{N}, \quad \overline{n}_j \equiv \frac{\xi_j \overline{N}_j}{N}.$$

Substituting equation (6.4.14) into equation (A.6.4.3), we solve

$$\overline{k}_1 = \Lambda\left(k_i, \overline{k}_2\right) \equiv \frac{k_i f_s^*(k_i) - \overline{\Omega}(k_i) - \left(\overline{\Omega}_0(k_i)\overline{n}_2 + f_s^*(k_i)\overline{N}_2/N\right)\overline{k}_2}{\overline{\Omega}_0(k_i)\overline{n}_1 + f_s(\Omega) f_i'(k_i)\overline{N}_1 / f_s^{*\prime}(k_i) N}, \tag{A.6.4.4}$$

where

$$f_s^{*\prime}(k_i) = \frac{f_i'(k_i) f_s'(\Omega)\Omega' + f_s(\Omega) f_i''(k_i)}{f_s'(\Omega)} - \frac{f_s(\Omega) f_i'(k_i) f_s''(\Omega)\Omega'}{f_s'^2(\Omega)}.$$

Insert $s_j = \lambda_j \hat{y}_j$ and equations (6.4.4) into (6.4.9)

$$\dot{\overline{k}}_j = \left(\lambda_j r - \xi_j\right)\overline{k}_j + \lambda_j w_j.$$

Substitute equation (A.6.4.4) and r and w_j in equations (6.4.2) and (6.4.3) into the above equations

$$\dot{\overline{k}}_1 = \Lambda_1\left(k_i, \overline{k}_2\right) \equiv \left(f_i'(k_i) - \delta_1\right)\lambda_1 \Lambda\left(k_i, \overline{k}_2\right) + \lambda_1 h_1\left(f_i(k_i) - k_i f_i'(k_i)\right),$$

$$\dot{\overline{k}}_2 = \Lambda_2\left(k_i, \overline{k}_2\right) \equiv \left(f_i'(k_i) - \delta_2\right)\lambda_2 \overline{k}_2 + \lambda_2 h_2\left(f_i(k_i) - k_i f_i'(k_i)\right), \quad \text{(A.6.4.5)}$$

where $\delta_j \equiv \delta_k + \xi_j/\lambda_j$. The above 2 differential equations contain three variables, $\overline{k}_1, \overline{k}_2$, and k_i. Equation (A.6.4.4) relates $\overline{k}_1$ to $\overline{k}_2$ and k_i. Take derivatives of equation (A.6.4.4) with respect to t

$$\dot{\overline{k}}_1 = \Lambda_i\left(k_i, \overline{k}_2\right)\dot{\overline{k}}_i - \frac{\left(\overline{\Omega}_0 \overline{n}_2 + f_s^*(k_i)\overline{N}_2/N\right)}{\overline{\Omega}_0 \overline{n}_1 + f_s^*(k_i)\overline{N}_1/N}\Lambda_2\left(k_i, \overline{k}_2\right), \quad \text{(A.6.4.6)}$$

where

$$\Lambda_i\left(k_i, \overline{k}_2\right)$$

$$\equiv \left[\frac{f_s^* + k_i f_s^{*\prime}\Omega' - \overline{\Omega}' - \left(\overline{\Omega}_0' \overline{n}_2 + f_s^{*\prime}\Omega' \overline{N}_2/N\right)\overline{k}_2 - \left(\overline{\Omega}_0' \overline{n}_1 + f_s^{*\prime}\Omega' \overline{N}_1/N\right)}{\overline{\Omega}_0 \overline{n}_1 + f_s^* \overline{N}_1/N}\right],$$

$$\overline{\Omega}_0' = \left(1 - \Omega'\right)\left(f' + \delta\right) + (k_i - \Omega) f'',$$

$$\overline{\Omega}'(k_i) = \left[\left(1 - \Omega'\right)\left(f - k_i f'\right) - (k_i - \Omega) k_i f''\right]\frac{\sum_j h_j \xi_j \overline{N}_j}{N}.$$

Substituting equation (A.6.4.6) into the dynamic system (A.6.4.5), we obtain

$$\dot{\overline{k}}_i = \frac{\Lambda_1\left(k_i, \overline{k}_2\right)}{\Lambda_i\left(k_i, \overline{k}_2\right)} + \frac{\left(\overline{\Omega}_0 \overline{n}_2 + f_s^* \overline{N}_2/N\right)}{\overline{\Omega}_0 \overline{n}_1 + f_s^* \overline{N}_1/N}\frac{\Lambda_2\left(k_i, \overline{k}_2\right)}{\Lambda_i\left(k_i, \overline{k}_2\right)},$$

$$\dot{\overline{k}}_2 = \Lambda_2\left(k_i, \overline{k}_2\right).$$

The two differential equations contain two variables. With proper initial conditions $k_i(t)$ and $\overline{k}_2(t)$, the above equations determine values of $k_i(t)$ and $\overline{k}_2(t)$ over time.

7 Monetary growth with urban structure

Uneven distribution of economic and social activities over time and space is a well-observed phenomenon. Irrespective of recent rapid developments of technology for transportation and communications, agglomeration of economic activities is still a main character of modern economic systems. As pointed out by Henderson *et al.* (2001), a proper understanding of spatial economics is essential for explaining economic growth. Over the past three decades, the study of economic growth with housing and economic geography has caused a lot of attention in urban economics and regional science.[1] Yet, we argue that urban economics still needs an analytical framework for spatial evolution and growth with capital accumulation. As pointed out by Baldwin and Martin (2004: 2675–6),

> Many of the most popular economic geography models focus on labor. ... These are unsuited to the study of growth. The key to all sustained growth is the accumulation of human capital, physical capital and/or knowledge capital – with the accumulation of knowledge capital, i.e. technological progress having a privileged position. We thus need a model in which capital exists and its stock is endogenous.

As early as 1980, Arnott (1980: 53) pointed out:

> In the last decade the static theory of residential urban location and land use has been extensively developed. The theory has generated many useful insights, but because it ignores growth and durability of housing and urban infrastructure there are many urban phenomena it cannot explain.

This description is still applicable to (most of) the current literature of urban economics. Fujita and Thisse (2002a: 389) state the current situations as follows:

> Clearly, space and time are intrinsically mixed in the process of economic development. However, the study of their interaction is a formidable task. Because either agglomeration or growth is a complex phenomenon by itself, one should expect any integrated analysis to face many conceptual and

analytical hurdles. Not surprisingly, therefore, the field is still in its infancy, and relevant contributions have been few.

The necessity of analyzing urban configuration and economic growth as a connected whole has long been recognized.[2] The purpose of this chapter is to introduce money into the OSG model with capital accumulation and the standard land-use model in urban economics and regional science. We synthesize the main ideas in the three key models within a comprehensive framework. The three models are the Solow growth model, the Alonso urban model and the Muth housing model. Numerous contributions to urban economics have followed the equilibrium theory of urban land market pioneered by Alonso (1964). This approach has been extended in many directions. The earlier important contributions are made by Muth (1969), Mills (1967), Beckmann (1969), Solow (1972), and others. There are many contributions to the literature in the last two decades. However, it is argued that one of the limitations of urban land-use models is that most of the models concentrate on the residential location and urban structure and neglect production aspects of urban dynamics. On the other hand, urban growth theory is another important theoretical approach to explain spatial phenomena. There are many studies of urban growth and dynamics.[3] The development of this approach has been influenced by growth theory and theory of international trade. From the point of view of economic geography, one of the limitations in most of the urban growth models is that they omit urban land use. We see that urban economics studies economic geography with wages and capital accumulation fixed and urban growth theory concentrates on growth factors such as capital accumulation, omitting spatial structures of urban systems. We will develop a dynamic urban growth model in which the main ideas in the two approaches are synthesized in a comprehensive framework.

The study of housing investment and prices is one of the key issues in regional science and urban economics. Housing is the largest component of nonhuman wealth for households and housing services are a fundamental component of the household consumption. Housing is closely related to the rest of the economy. For instance, in the United States, real estate investment accounts for over 50 percent of total private investment and real estate assets represent just under 60 percent of the nation's wealth. Almost 70 per cent of U.S. real estate is residential.[4] Housing has a set of intrinsic properties, which make it significantly different from any other goods. Operation of the housing market is varied in different regions and countries. There is a large amount of literature on durable housing in spatial context.[5] Nevertheless, as argued by Brito and Pereira (2002), the link between the housing market and long-term growth has been neglected in the literature. This study tries to contribute to the issue.

The objective of this chapter is to study monetary growth with economic geography. We are concerned with the residential land use pattern and determination of goods production, capital accumulation, housing price, land rent over time and space. The model is a one-dimensional model of residential location

with a central business center. The chapter accepts the assumption that the city has centered employment. The urban structure with a single CBD is apparently not what we often observe in reality. As shown by Wheaton (2004), in U.S. cities actual employment is almost as dispersed as residences. Land should have "mixed" rather than exclusive use in general. We use the linear monocentric city as this allows us to explicitly analyze the model.

7.1 The monetary urban growth model

The model is a combination of the basic features of the three key models, the Solow growth model, the Alonso urban model, and the Muth housing model in the neoclassical growth theory and urban economics. Although the model is based on Zhang (1993b), the model is improved in many aspects. The production function in this section is of a general form; while in the previous work the function is of the Cobb-Douglas form. The section assumes the equally-shared landownership; while the previous study assumes the absentee land ownership. This study analyzes equilibrium and simulates the model over time, while the previous study is only concerned with the equilibrium point. Moreover, this study provides a rational mechanism of locational choice, while the previous study lacks a utility-maximization mechanism for the decision. As far as urban structures are concerned, we follow the standard residential land-use model. The basic features of this model are that an isolated city state is built on a flat featureless plain. All residents in the economy work in the CBD. People travel only between their homes and the CBD. Travel is equally costly in terms of time or/and money in all directions. An individual may reside at only one location. The only spatial characteristic of any location that directly matters is the distance from the city center. Due to the complexity of taking both space and time in a general equilibrium framework, at this initial stage we simplify the structure of the isolated economic system. The population is homogenous.[6] The households achieve the same utility level regardless of where they locate. All the markets are perfectly competitive. The system is geographically linear and consists of two parts – the CBD and the residential area. The isolated state consists of a finite strip of land extending from a fixed CBD with constant unit width. We assume that all economic activities are concentrated in the CBD. We neglect public goods. The households occupy the residential area. We assume that the CBD is located at the left-side end of the linear territory. As we will get the same conclusions if we locate the CBD at the center of the linear system, the specified urban configuration will not affect our discussion.

The system consists of two sectors: industrial and housing, sectors. The industrial production is the same as that in the one-sector neoclassical growth model. We assume that the industrial product can be either invested or consumed. The housing production is similar to that in the Muth model. Housing is supplied with a combination of capital and land. We assume that the total labor force is fully employed by the industrial sector. We select industrial goods to serve

as numeraire. As we assume that the transportation cost of workers to the city is dependent on the travel distance, land rent for housing should be spatially different.

First, we introduce:

N: the fixed population;
$K_i(t)$: the capital stocks employed by the industrial sector at time t;
$w(t)$ and $r(t)$: the wage rate and the rate of interest, respectively; and
$F(t)$ and $C(t)$: the output of the industrial sector and the total consumption of the commodity, respectively.

We assume that industrial production is carried out by a combination of capital and labor force in the form of $F(K_i(t), N)$, where F is the output of the industrial sector. Assume F to be neoclassical. Markets are competitive; thus labor and capital earn their marginal products, and firms earn zero profits. The rate of interest, $r(t)$, and wage rates, $w(t)$, are determined by markets. Hence, for any individual firm $r(t)$ and $w(t)$ are given at each point of time. The production sector chooses the two variables, $K_i(t)$ and N, to maximize its profit. The marginal conditions are given by

$$r + \delta_k = F_{K_i}(K_i, N), \quad w(t) = F_N(K_i, N), \tag{7.1.1}$$

where δ_k is the depreciation rate of physical capital.

We now describe housing production and behavior of households. First, we introduce:

L: the fixed (territory) length of the isolated state;
ω: the distance from the CBD to a point in the residential area;
$R(\omega, t)$ and $R_h(\omega, t)$: the land rent and housing rent per household at location ω;
$k(\omega, t)$ and $s(\omega, t)$: the wealth owned by and the saving made by the household at location ω, respectively;
$c(\omega, t)$: the consumption of the household at location ω;
$n(\omega, t)$ and $L_h(\omega, t)$: the residential density and the lot size of the household at location ω;
$K_h(t)$: the capital stocks employed by the housing sector; and
$K(t)$: the total capital stock of the economy.

We assume that all housing is residential housing. Residential housing assets play a dual role in the economy. First, residential housing assets are used as a durable consumption good. They are the source of housing services. Residential housing assets are used as a mechanism for the intertemporal transfer of wealth, which generates both rents and capital gains through housing appreciation. We limit ourselves to a simple case of housing technology. To explain space-dependent gradients for residential density and capital–land ratios, Muth introduces a

commodity "housing" rather than land in describing dwelling conditions. Housing is produced with land and non-land inputs. Households have a derived demand for land, dependent on both preferences for housing and technical characteristics of housing production function. We follow this approach in explaining decision making of dwelling sites of households. The housing industry supplies housing services by combining land and capital. Let us denote $c_h(\omega, t)$ housing service received by the household at location ω. We specify the housing service production function as follows

$$c_h(\omega, t) = k_h^{\alpha_h}(\omega, t) L_h^{\beta_h}(\omega, t), \quad \alpha_h + \beta_h = 1, \quad \alpha_h, \beta_h \geq 0, \tag{7.1.2}$$

where $k_h(\omega, t)$ is the input level of capital per household at location ω. Here, we assume that housing capital can be instantaneously adjusted. All the characteristics of houses such as the size of a lot or the size of a house can be changed instantaneously without costs. Hence, the capital–land ratio is always perfectly adjusted.[7]

The marginal conditions are given by

$$r + \delta_k = \frac{\alpha_h R_h c_h}{k_h}, \quad R = \frac{\beta_h R_h c_h}{L_h}, \quad 0 \leq \omega \leq L. \tag{7.1.3}$$

According to the definitions of L_h and n, we have

$$n(\omega, t) = \frac{1}{L_h(\omega, t)}, \quad 0 \leq \omega \leq L. \tag{7.1.4}$$

The total capital stocks employed by the housing sector are equal to the sum of the capital stocks for housing over space at any point of time. The relationship between $k_h(\omega, t)$ and $K_h(t)$ is thus given by

$$K_h(t) = \int_0^L n(\omega, t) k_h(\omega, t) d\omega. \tag{7.1.5}$$

Each worker may get income from land ownership, wealth ownership and wages. In order to define incomes, it is necessary to determine land ownership structure. It can be seen that land properties may be distributed in multiple ways under various institutions. To simplify the model, we assume the land is equally owned by the population. This implies that the revenue from land is equally shared among the population.[8] The total land revenue, $\overline{R}(t)$, is given by

$$\overline{R}(t) = \int_0^L R(\omega, t) d\omega.$$

The income from land per household, $\bar{r}(t)$, is given by

$$\bar{r}(t) = \frac{\overline{R}(t)}{N}.$$

Let $P(t)$, μ, $M(t)$, and $m(t)$ be defined as in Section 2.2. With μ being the constant net growth rate of the per-capita money stock, $M(t)$, evolves over time according to

$$\dot{M}(t) = \mu M(t), \quad \mu > 0.$$

The government expenditure in real terms per capita, $\tau(t)$, is given by

$$\tau(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\mu M(t)}{P(t)} = \mu m(t).$$

The representative household receives $\mu m(t)$ units of paper money from the government through a "helicopter drop," also considered independent of his money holdings. Consumers make decisions based on choice of lot size, consumption level of commodity as well as on how much to save. Let $\overline{m}(\omega, t)$ and $k(\omega, t)$ stand for respectively the per capita real money balances and wealth (excluding land) owned by the typical household at location ω. Each household at ω obtains the real income

$$y(\omega,t)=r(t)k(\omega,t)+w(t)-\pi(t)\overline{m}(\omega,t)+\mu m(t)+\bar{r}(t), \quad 0\leq\omega\leq L. \quad (7.1.6)$$

The disposable income is given by

$$\hat{y}(\omega,t)=y(\omega,t)+a(\omega,t), \quad (7.1.7)$$

where

$$a(\omega,t)\equiv k(\omega,t)+\overline{m}(\omega,t).$$

In this chapter, we assume that urban land use is concentrated on residential use. Actually, production and transportation use land. For instance, Solow and Vickery (1971) examine use of urban land for transportation in a similar spatial setting as in this study. We may consider that at each location, land is used either for transportation and housing as follows[9]

$$L_H(\omega)+L_T(\omega)\leq 1$$

where $L_H(\omega)$ and $L_T(\omega)$ are respectively the ratios of land devoted to housing and transportation. We may thus treat travel time to the CBD at ω as a function of the residential density and transportation conditions from the CBD to the location.

At each point of time, a consumer at location ω distributes the total available budget among the leisure time, $T_h(\omega)$, housing, $c_h(\omega,t)$, saving, $s(\omega,t)$, and consumption of industrial goods, $c(\omega,t)$. Here, we assume that the leisure is only dependent on the residential location as the work time is fixed and equal for each household, in disregard of residential location. After the work time is decided, the households decide the time distribution between leisure and travel to work. As we assume that the travel time from the CBD to the residential location is only related to the distance and neglect any other effects such as technological change, infrastructure improvement, and congestion on the travel time from the CBD to the residential area, the leisure time, which is equal to the fixed total time minus the travel time, is only related to location. Let T_0 and $\Gamma(\omega)$ respectively stand for the total available time for travel and leisure and the time spent on traveling between the residence and CBD. We have

$$T_h(\omega) = T_0 - \Gamma(\omega).$$

The budget constraint is given by

$$(1+r(t))\overline{m}(\omega,t)+R_h(\omega,t)c_h(\omega,t)+c(\omega,t)+s(\omega,t)=\hat{y}(\omega,t).$$

Equation (7.1.8) means that the consumption and saving exhaust the consumers' disposable personal income. Substituting equation (7.1.8) into the budget constraint yields

$$(r(t)+\pi(t))\overline{m}(\omega,t)+R_h(\omega,t)c_h(\omega,t)+c(\omega,t)+s(\omega,t)=\bar{y}(\omega,t), \tag{7.1.8}$$

where

$$\bar{y}(\omega,t)\equiv(1+r(t))k(\omega,t)+w(t)+\mu m(t)+\bar{r}(t).$$

Location choice is closely related to the existence and quality of such physical environmental attributes as open space and noise pollution as well as social environmental quality. We assume that utility level, $U(\omega,t)$, of the household at location ω is dependent on $\overline{m}(\omega,t), T_h(\omega), c_h(\omega,t), s(\omega,t)$, and $c(\omega,t)$ as follows

$$U(\omega,t)=\theta(\omega,t)\overline{m}^{\varepsilon_0}(\omega,t)T_h^{\sigma_0}(\omega)c^{\xi_0}(\omega,t)c_h^{\eta_0}(\omega,t)s^{\lambda_0}(\omega,t),$$
$$\varepsilon_0,\sigma_0,\xi_0,\eta_0,\lambda_0>0, \tag{7.1.9}$$

in which $\varepsilon_0,\sigma_0,\xi_0,\eta_0$, and λ_0 are a typical person's elasticity of utility of real money balance, leisure time, industrial goods, housing, and saving at ω. We call $\varepsilon_0,\sigma_0,\xi_0,\eta_0$, and λ_0 propensities to hold real money balance, to use leisure time, to consume goods, to consume housing, and to hold wealth, respectively.

Zhang (1993c) first introduces amenity into an urban equilibrium model. As argued by Glaeser *et al.* (2001), consumption amenities have increasingly played a more important role in urban formation. Public services, accessibilities, local transportation systems, pollution, and human relations such as discrimination all involve externalities and affect amenities. We incorporate amenity into the consumer location decision by assuming that amenity is a function of residential density.[10] Distance from the CBD reflects two elements: the inconvenience of the distance and the value of the amenity of the surrounding area. The urban dynamics is influenced by many changing characteristics of environmental quality such as air quality, levels of noise pollution, open space, and other physical and social neighborhood qualities at each location. Environmental quality can be reflected in part by its effect on the location choice of the individual. Many kinds of externalities may actually exist at any location. Some may be historically given, such as historical buildings and climate; others such as noise and cleanliness, may be endogenously determined by the location of residents. Households may prefer a low-density residential area to a high one, as this tends to have more green space, less noise, cleaner and more safety in a low-density area. Nevertheless, there are other factors, such as social interactions, which may make high-density areas attractive. Glaeser *et al.* (2001: 27) point out: "Most urban scholars think of cities as offering positive agglomeration benefits in the productive sphere, and as having negative agglomeration effects (or congestion effects) on non-work consumption."[11] We consider that residential densities may have positive or negative agglomeration effects. We specify the amenity, $\theta(\omega, t)$, at ω as follows

$$\theta(\omega, t) = \theta_1 n^{\vartheta}(\omega, t), \quad \theta_1 > 0.$$

The function, $\theta(\omega, t)$, implies that the amenity level at location ω is related to the residential density at the location.[12]

For simplicity, we restrict the analysis to the case that all households obtain the same level of utility at any point of time. This also comes out of our assumption that the population is homogeneous and people can change their residential location freely without any transaction costs and time delay. The conditions that households get the same level of utility at any location at each point of time is represented by

$$U(\omega_1, t) = U(\omega_2, t), \quad 0 \leq \omega_1, \quad \omega_2 \leq L.$$

It should be noted that we neglect possible costs for migration. In reality, to change housing location costs. Although the condition of utility equalization is often used in the literature of urban economics, the assumption of utility equalization is rarely used in the literature of economic dynamics as the temporary equilibrium condition of population distribution.

Maximizing $U(\omega, t)$ subject to the budget constraint (7.1.8) yields

$$\overline{m}(\omega,t)=\frac{\varepsilon\overline{y}(\omega,t)}{r(t)+\pi(t)},\ c(\omega,t)=\xi\overline{y}(\omega,t),c_h(\omega,t)=\frac{\eta\overline{y}(\omega,t)}{R_h(\omega,t)},$$
$$s(\omega,t)=\lambda\overline{y}(\omega,t), \tag{7.1.10}$$

where

$$\varepsilon\equiv\rho\varepsilon_0,\quad \sigma\equiv\rho\sigma_0,\quad \xi\equiv\rho\xi_0,\quad \eta\equiv\rho\eta_0,\quad \lambda\equiv\rho\lambda_0,$$
$$\rho\equiv\frac{1}{\varepsilon_0+\sigma_0+\xi_0+\eta_0+\lambda_0}.$$

The above equations mean that the real money balance, housing consumption, consumption of the good, and saving are positively proportional to the available income.

According to the definition of $s(\omega, t)$, the capital accumulation for the household at location ω is given by

$$\dot{a}(\omega,t)=s(\omega,t)-a(\omega,t),\quad 0\le\omega\le L. \tag{7.1.11}$$

As the state is isolated, the total population is distributed over the whole urban area. The population constraint is given by

$$\int_0^L n(\omega,t)d\omega=N. \tag{7.1.12}$$

The total consumption, $C(t)$, is given by

$$\int_0^L n(\omega,t)c(\omega,t)d\omega=C(t). \tag{7.1.13}$$

The national real balance is equal to the sum of the real money balances of all the households. That is[13]

$$\int_0^L \overline{m}(\omega,t)n(\omega,t)d\omega=Nm(t). \tag{7.1.14}$$

The assumption that capital is fully employed is given by

$$K_i(t)+K_h(t)=K(t). \tag{7.1.15}$$

The total capital stocks employed by the production sectors are equal to the total wealth owned by all the households. That is,

$$\int_0^L k(\omega, t)n(\omega, t)d\omega = K(t). \tag{7.1.16}$$

We have thus built the dynamic growth model with endogenous spatial distribution of real money balance, wealth, consumption and population, capital accumulation, and residential location. We now examine dynamic properties of the system.

7.2 The dynamics of the non-monetary economy

As analysis of the monetary system is complicated, we are first concerned with a special case when there is no money in the system, that is, $\varepsilon_0 = 0$.

7.2.1 The motion of the system and steady state

In Appendix 7.1, we show that when $\varepsilon_0 = 0$, the dynamics can be expressed by a single differential equation with the variable, $K_i(t)$. We omit ω or/and t in the expressions, wherever without causing confusion.

Lemma 7.2.1
The dynamics of the national wealth is determined by

$$\dot{K}_i = \frac{\lambda F_{K_i} F - (\xi + \lambda\delta_k)F_{K_i}K_i - (\xi + \lambda)\bar{\alpha}F}{(\xi + \lambda)\left(F_{K_i} - \delta\bar{\alpha}\right)\Lambda^*(K_i)}, \tag{7.2.1}$$

in which $\Lambda^*(K_i) > 0$ is a function of K_i explicitly defined in Appendix 7.1. At any point of time, all the other variables are determined as a unique function of $K_i(t)$ by the following procedure: K by (A.7.1.19) $\rightarrow K_h = K - K_i \rightarrow F(K_i, N) \rightarrow r$ and w by (7.1.1) $\rightarrow C$ and S by (A.7.1.1) $\rightarrow \bar{r}$ by (A.7.1.7) $\rightarrow k(\omega)$ by (A.7.1.9) $\rightarrow \hat{y}(\omega)$ by (A.7.1.8) $\rightarrow k_h(\omega)$ by (A.7.1.12) $\rightarrow n(\omega)$ by (A.7.1.14) and (A.7.1.15) $\rightarrow R_h(\omega)$ by (A.7.1.13) $\rightarrow L_h(\omega) = 1/n(\omega) \rightarrow c_h(\omega) = k_h^{\alpha_h}(\omega)L_h^{\beta_h}(\omega) \rightarrow R(\omega)$ by (7.1.3) $\rightarrow c(\omega)$ and $s(\omega)$ by (7.1.10).

This lemma implies that once we solve the one-dimensional differential equation (7.2.1), then we can determine all the other variables, such as the national output, national housing product, capital distribution between the two sectors, the rate of interest, the wage rate, and the income from the land ownership, the income and wealth distribution over space, residential distribution, the total transportation time spent in the system, the leisure time distribution and the total leisure time in the economy, land rent distribution, and housing rent distribution.

It should be noted that the total travel time, $\tilde{T}$, and the total leisure time, $\tilde{T}_h$, are given by

$$\tilde{T} = \int_0^L n(\omega)\Gamma(\omega)d\omega, \quad \tilde{T}_h = \int_0^L n(\omega)(T_0 - \Gamma(\omega))d\omega.$$

As proved in the Appendix, if the initial distribution of the wealth is homogenous over space, then the per-capita wealth, $k(\omega)$, and disposable income, $\hat{y}(\omega)$, are homogenous over space. That is

$$k(\omega_1, t) = k(\omega_2, t), \quad \hat{y}(\omega_1, t) = \hat{y}(\omega_2, t), \quad 0 \leq \omega_1, \omega_2 \leq L.$$

This implies that all residents receive the same income and own the same amount of wealth at each point of time. As shown in the Appendix, this property results from the assumption that we neglect transportation costs and the specified landownership structure. If possible transportation costs are not omitted or/and some other type of landownership are assumed, the property may not hold.[14] We see that the omission of transportation costs greatly simplifies our dynamic analysis. This is the reason we omit transportation costs and take account of travel time in introducing distance effects in our model. As far as economic geography is concerned, the omission of transportation costs may not affect the main results of this model, except that the gradients of the residential density curve and land rent may be steeper when transportation costs are taken into account.

The residential distribution is given by equations (A.7.1.14) and (A.7.1.15). That is

$$\frac{n(\omega_1)}{n(\omega_2)} = \left(\frac{T_h(\omega_1)}{T_h(\omega_2)}\right)^{\beta_0}, \quad n(0) = \frac{NT_h^{\beta_0}(0)}{\int_0^L T_h^{\beta_0}(\omega)d\omega}, \quad 0 \leq \omega_1, \omega_2 \leq L,$$

where $\beta_0 = \sigma/(\eta\beta_n - \mu)$. The residential density is invariant over time. The conclusion comes from the assumptions that the population fixed, the land is equally shared among the residents, there are no congestion, no transportation costs, no investment in the transportation infrastructure, and no technological change in transportation systems. If any of these assumptions is relaxed, then $n(\omega)$ may become time-dependent. For instance, if we introduce some (exogenous) technological change on the transportation system,[15] then the residential distribution is not invariant in time.

We now examine changes of the residential density over space. As $T_h(\omega_1)/T_h(\omega_2) > 1$ if $\omega_1 < \omega_2$, we see that if $\beta_0 > 0$, then $n(\omega_1) > n(\omega_2)$. The residential density is higher nearer the CBD. As traveling costs more as the residence is further away from the CBD, it is reasonable to observe the declination of the residential density as the distance from the CBD increases. Yet, if $\beta_0 < 0$,the residential

density distribution is inverse to the case of $\beta_0 > 0$ in the sense that the further the residential location is away from the CBD, the higher the density is. To see why this happens, from the definition of β_0 we see that the parameter becomes negative only when $\mu > \eta\beta_n > 0$. A positive μ means that as the residential density rises, the local attractiveness as residential area is increased. The condition, $\mu > \eta\beta_n$, implies that the individual welfare is positively strongly affected by, for instance, social interactions among local people and the propensity to use land is low.[16] As people so highly evaluate interacting with each other among locals, the urban residential density becomes higher further away from the CBD. In this study, we limit our examination to the case of $\mu < \eta\beta_n$.[17]

If we interpret the amenity only related to the social interactions, we see that social interactions impact shapes of the residential distribution. It is well known that Beckmann (1976) first proposed an urban residential model with social interactions. In his approach, the utility of an individual depends on the average distance to all individuals with whom this person interacts as well as on the amount of land the individual buys on the market. In the Beckmann model, the residential center is endogenously formed. If we introduce Beckmann's approach, we may have that the highest density will neither appear at the CBD or at the opposite side of the CBD; instead the highest density may appear somewhere between.[18] If we introduce more realistic forms of amenity functions and take account of transportation costs, rather than being monotonic with respect to distance, possible urban residential shapes will become far more complicated. As urban systems with growth and spatial structures are so complicated, even a slight change in assumptions might cause great analytical difficulties.

By equation (7.2.2), we have

$$\frac{R_h(\omega_1)}{R_h(\omega_2)} = \left(\frac{n(\omega_1)}{n(\omega_2)}\right)^{\beta_h}.$$

From equation (7.2.2), we see that $R_h(\omega, t)$ is changeable with time; the above equation tells that the ratio of housing rents between any two locations is invariant time. As $n(\omega_1) > n(\omega_2)$, we see that the housing rent declines as the residential area is further away from the CBD. By equations (7.1.10), we have

$$c(\omega_1) = c(\omega_2),\ \frac{c_h(\omega_1)}{c_h(\omega_2)} = \frac{R_h(\omega_2)}{R_h(\omega_1)},\ s(\omega_1) = s(\omega_2).$$

The consumption of industrial goods is invariant in location but changeable in time. This property comes from that the households have the same income, the same propensity to consume goods, and the price of the goods are invariant in space. Consumption of housing increases as the residential area is further from the CBD. It should be noted that the total expenditure on housing by per household, $R_h(\omega, t)c_h(\omega, t)$, is invariant in space but changeable in time. That is

$$R_h(\omega_1, t)c_h(\omega_1, t) = R_h(\omega_2, t)c_h(\omega_2, t).$$

This property comes from that the households have the same income and the same propensity to consume housing. As the housing rent declines in location, we see that the housing consumption per household increases as the dwelling site is further away from the CBD.

So far, our analyses have been conducted with the general form of the production function. Although we can analyze all the properties of the dynamic system, it is not easy to explicitly express and interpret the analytical results. Moreover, for simulation we also need to specify the form of the production function. In the next sections, we explain dynamic properties by specifying the production function in the Cobb-Douglas form

$$F = A_i K_i^{\alpha} N^{\beta}, \quad \alpha + \beta = 1, \quad \alpha, \beta > 0. \tag{7.2.2}$$

Introduce $A \equiv A_i N^{\beta}$. By equation (A.7.1.19) we solve

$$K = \frac{(\alpha + \overline{\alpha}) K_i F}{\alpha F - \delta \overline{\alpha} K_i}, \tag{7.2.3}$$

in which $\delta = 1 - \delta_k$ and $\overline{\alpha} \equiv \alpha_h \eta / (\xi + \lambda)$. By equations (7.2.1) and (A.7.1.20), we get

$$\dot{K}_i = \frac{\lambda F_{K_i} - (\xi + \lambda \delta_k)\alpha - (\xi + \lambda)\overline{\alpha}}{(\xi + \lambda)\left(F_{K_i} - \delta \overline{\alpha}\right) \Lambda^*(K_i)} F, \tag{7.2.4}$$

in which

$$\Lambda^*(K_i) = \alpha F \left\{ \frac{(\alpha F - \delta \overline{\alpha} K_i)(1 + \overline{\alpha}) + (\delta K_i + F)\beta \overline{\alpha}}{(\alpha F - \delta \overline{\alpha} K_i)^2} \right\} > 0.$$

It is straightforward to prove that the one-dimensional differential equation (7.2.4) has a unique equilibrium point

$$K_i = \left\{ \frac{\alpha \lambda A}{(\xi + \lambda \delta_k)\alpha + (\xi + \lambda)\overline{\alpha}} \right\}^{1/\beta}. \tag{7.2.5}$$

Moreover, the unique equilibrium is stable.

According to Lemma 1, for $K_i(t)$ determined by differential equation (7.2.4) we can determine all the other variables as functions of $K_i(t)$. We now solve these variables. First, by equations (7.2.2) and (7.2.3), we get $F(t)$ and $K(t)$. By equation (A.7.1.2), we have

$$\hat{Y} = \frac{F + \delta K}{\xi + \lambda}. \tag{7.2.6}$$

By equations (7.1.1) and (7.1.16), we have

$$r = \frac{\alpha F}{K_i} - \delta_k, \quad w = \frac{\beta F}{N}, \quad K_h = K - K_i. \tag{7.2.7}$$

We solve C and S by equation (A.7.1.1), $\bar{r}$ by equation (A.7.1.7), $k(\omega) = K/N$, $\hat{y}(\omega)$ by equation (A.7.1.8) $k_h(\omega)$ by equation (A.7.1.12). To solve $n(\omega)$ by equations (A.7.1.14) and (A.7.1.15), we specify the travel time as a linear function of ω by $\upsilon\omega$, where $1/\upsilon$ is the speed of the travel. The leisure time is thus given by

$$T_h(\omega) = 1 - \upsilon\omega, \quad 0 \leq \omega \leq L,$$

where we specify $T_0 = 1$. By equations (A.7.1.14) and (A.7.1.15), we solve

$$n(\omega) = n(0)\, T_h^{\beta_0}(\omega), \quad n(0) = \frac{(1+\beta_0)\upsilon N}{1-(1-\upsilon L)^{1+\beta_0}}, \tag{7.2.8}$$

where we use $T_h(0) = 1$. We solve $R_h(\omega)$ by equation (A.7.1.13), $L_h(\omega) = 1/n(\omega)$, $c_h(\omega) = k_h^{\alpha_h}(\omega) L_h^{\beta_h}(\omega)$, $R(\omega)$ by equation (7.1.3), and $c(\omega)$ and $s(\omega)$ by equations (7.1.10).

Theorem 7.2.1
If the production function is specified in the Cobb-Douglas form, the urban dynamic system has a unique stable equilibrium point. All the variables are explicitly solved by the procedure given above.

As we have proved that the system has a unique stable equilibrium and explicitly solved the location-related variables, it is straightforward to examine effects of change in any parameter on the equilibrium of the urban system. Before examining changes in the consumers' propensity to save on the economic structure, we explain the capital–land ratio in the housing market.

It is observed that as we get closer to the city center, not only the population density but also the capital intensity per square miles increase. Buildings tend to be higher near urban buildings. We now explain this phenomenon. The capital–land ratio, $k_h(\omega, t)/L_h(\omega, t)$, is given by

$$\frac{k_h(\omega,t)}{L_h(\omega,t)} = k_h(\omega,t)\, n(\omega,t) = \frac{\alpha_0 \hat{y}(t) T_h^{\beta_0}(\omega)}{r(t)+\delta_k},$$

where we use equations (A.7.1.12), (A.7.1.14), and (A.7.1.15) and

$$\alpha_0 \equiv \frac{\alpha_h \eta N}{\int_0^L T_h^{\beta_0}(\omega)\, d\omega} > 0.$$

Corollary 7.2.1
At any point of time, the capital–land ratio decreases in the distance from the CBD.[19]

7.2.2 Comparative statics analysis

We now examine changes in the productivity of the industrial sector on the equilibrium of the urban economic structure. Taking derivatives of equation (7.2.5)

with respect to A_i yields

$$\frac{dK_i}{dA_i} = \frac{K_i}{\beta A_i} > 0.$$

An increase in the productivity increases the capital stock employed by the industrial sector. By equation (A.7.1.18), we have: $F = \bar{\lambda}K$. We have

$$\frac{1}{K}\frac{dK}{dA_i} = \frac{\alpha}{K_i}\frac{dK_i}{dA_i} > 0.$$

The national wealth rises. By $\hat{Y} = K/\lambda$, and equations (7.2.5) and (7.2.7), we have

$$\frac{d\hat{Y}}{dA_i} = \frac{1}{\lambda}\frac{dK}{dA_i} > 0, \quad \frac{dr}{dA_i} = -\frac{\beta r}{K_i}\frac{dK_i}{dA_i} < 0, \quad \frac{dw}{dA_i} = \frac{\alpha w}{K_i}\frac{dK_i}{dA_i} > 0,$$

$$\frac{dK_h}{dA_i} = \left(\frac{\alpha K}{K_i} - 1\right)\frac{dK_i}{dA_i} = \left(\frac{(\alpha\bar{\alpha} - \beta)F + \delta\bar{\alpha}K_i}{\alpha F - \delta\bar{\alpha}K_i}\right)\frac{dK_i}{dA_i}.$$

The rate of interest falls and the wage rate rises. As $k = K/N$ and $\hat{y} = \hat{Y}/N$, we have

$$\frac{dk}{dA_i} = \frac{1}{N}\frac{dK}{dA_i} > 0, \quad \frac{d\hat{y}}{dA_i} = \frac{1}{N}\frac{d\hat{Y}}{dA_i} > 0.$$

The wealth per capita and the disposable income are increased. By equations (7.1.10), we have

$$\frac{dc}{dA_i} = \xi\frac{d\hat{y}}{dA_i} > 0, \quad \frac{ds}{dA_i} = \lambda\frac{d\hat{y}}{dA_i} > 0.$$

By equations (A.7.1.12), we have

$$\frac{1}{\alpha_h c_h(\omega)}\frac{dc_h(\omega)}{dA_i} = \frac{1}{k_h(\omega)}\frac{dk_h(\omega)}{dA_i} = -\frac{1}{r+\delta_k}\frac{dr}{dA_i} + \frac{1}{\hat{y}(\omega)}\frac{d\hat{y}(\omega)}{dA_i} > 0. \tag{7.2.9}$$

The consumption level of housing and per-capita capital used for supplying housing are increased. From $R = (r+\delta_k)\beta_h n k_h/\alpha_h$ and equation (A.7.1.12), we have

$$\frac{1}{R(\omega)}\frac{dR(\omega)}{dA_i} = \frac{1}{\hat{y}}\frac{d\hat{y}}{dA_i} > 0. \tag{7.2.10}$$

The technological change in the industrial sector brings about land rent increases over the urban area. As the land is fixed and technological progress brings about

increases in income and decreases in the rate of interest (or in cost of the capital for housing), it is reasonable to expect that the land rent rises. As the land rent at every location is increased with the same proportion and the residential distribution is not affected by the technology, we have

$$\frac{1}{\bar{r}}\frac{d\bar{r}}{dA_i} = \frac{1}{\hat{y}}\frac{d\hat{y}}{dA_i} > 0.$$

The increase in the income from the land rent is equally distributed among the households. From $c_h(\omega) = \eta\hat{y}(\omega)/R_h(\omega)$, we have

$$\frac{1}{R_h(\omega)}\frac{dR_h(\omega)}{dA_i} = \frac{1}{\hat{y}(\omega)}\frac{d\hat{y}(\omega)}{dA_i} - \frac{1}{c_h(\omega)}\frac{dc_h(\omega)}{dA_i} = \frac{\beta_h}{R(\omega)}\frac{dR(\omega)}{dA_i} + \frac{\alpha_h}{r+\delta_k}\frac{dr}{dA_i}.$$

The housing rent may rise or fall, according to whether the land rent rises more rapidly or slowly than the interest rate falls down.

We have examined the impact of the technological change on the equilibrium values of all the variables in the dynamic system. It is straightforward to examine effects of change in any parameter on the system. In the remainder of this study, we carry out simulations to illustrate the motion of the dynamic system. Nevertheless, comparative statics analysis does not show how the system achieves its equilibrium from any initial condition and how it achieves its new equilibrium when the old one is disturbed due to exogenous changes in, for instance, the preference and the population.

7.2.3 Equilibrium and comparative statics analysis by simulation

This section determines the equilibrium values of all the variables by simulation and examines effects of some parameters on the economic geography. We specify values of the parameters as follows

$$A_i = 1.4,\ N = 10, L = 5,\ \alpha = 0.3,\ \alpha_h = 0.5,\ \eta = 0.05,\ \xi = 0.07,$$
$$\sigma = 0.2,\ \mu = -0.2,\ T_0 = 1,\ \upsilon = 0.05,\ \delta_k = 0.05. \qquad (7.2.11)$$

The population is fixed at 10 units and the urban size is fixed by 5 units. The productivity is specified with 1.4. We specify α with 0.3 partly because some empirical studies with the Cobb-Douglas production function uses this value.[20] The propensities to consume goods and consume housing are respectively specified at 0.05 and 0.07, which implies that the ratio between the expenditures on housing and other goods and services is about 0.71. The propensity to use leisure is specified at 0.2. The amenity parameter, μ, is negative. This implies that the households prefer to live in an area with low residential distribution. The total available time is fixed at unit and $\upsilon = 0.05$ meaning that if the total travel time from the CBD to another end of the system uses up 25 percent of the total available time. The depreciation rate is specified at 0.05.

Following Lemma 1 under (7.2.11), we calculate the equilibrium values of the location-independent variables as follows

$$F = 30.692,\ r = 0.017,\ w = 2.148,\ K_i = 136.868,\ K_h = 100.049,$$
$$K = 236.917,\ k = 23.692,\ \bar{r} = 0.673,\ \hat{y} = 26.922,\ c = 1.885,$$
$$s = 23.692,\ n(0) = 2.253.$$

The share of the capital employed by the housing sector is much lower than the capital stocks employed by the industrial sector. The income from the land, as a whole, is not large. The current income mainly comes from the wage and interest payment of non-land wealth. We now plot the equilibrium values of the location-dependent variables as in Figure 7.2.1. The residential density declines and the housing consumption increases in the distance from the CBD to the residential site. Both the housing and land rent fall in the distance. The housing rent curve is steeper than the land rent.

We now examine effects of changes in some parameters on the location-dependent variables.[21] First, we examine effects of changes in the amenity parameter, μ. We allow it to change from -0.9 to -0.1. From Figure 7.2.2, we see that as the parameter changes from -0.9 to -0.1, the residential density, housing consumption, housing and land rent all become increasingly steeper. The residential amenity has a strong effect on the urban configuration in our model.

Another important parameter is the propensity to use leisure time. In Figure 7.2.3, we change the propensity to use leisure time from 0.01 to 1. As leisure time is less valued, the urban configuration becomes slower. If the leisure time is highly evaluated, like at $\sigma = 1$, then the residential density, housing consumption, housing rent, and land rent are much steeper.[22]

We now examine effects of transportation conditions on the economic geography. The transportation system is measured by the parameter, ν. We allow this parameter to vary from 0.01 to 0.06. As the transportation condition is improved, the urban configuration becomes flatter. The effects of improvement in the transportation system are similar to those in the change of the propensity to use leisure.

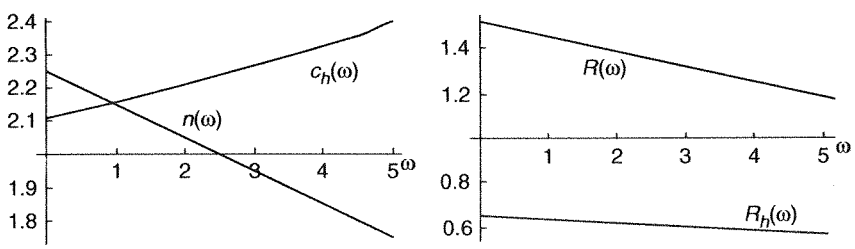


Figure 7.2.1 The equilibrium values of the location-dependent variables.

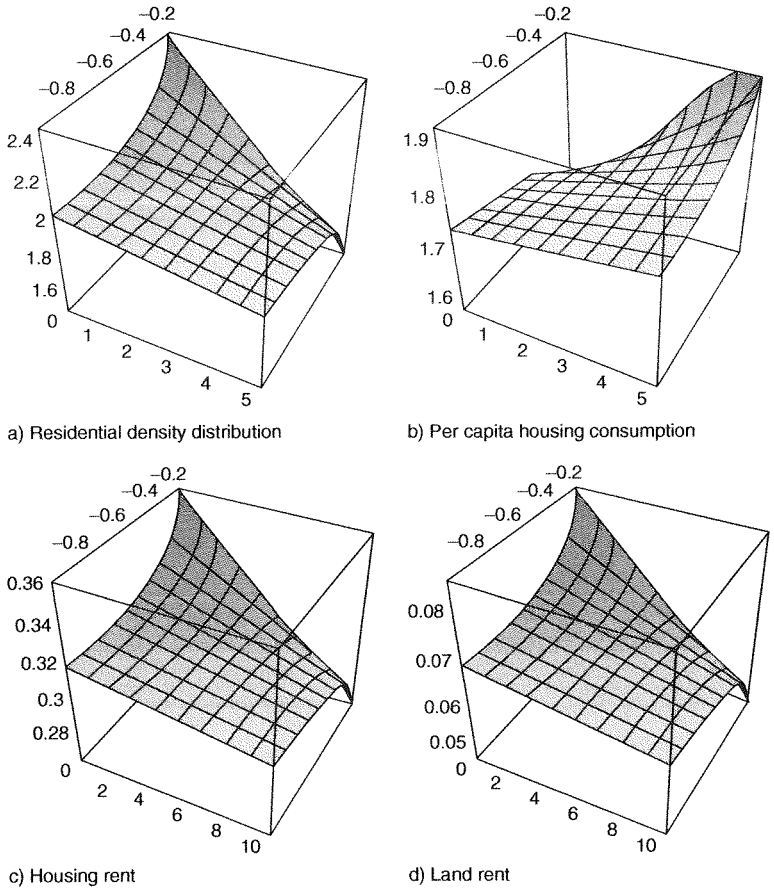


Figure 7.2.2 The economic geography and the amenity parameter.

7.2.4 Motion and comparative dynamic analysis by simulation

The previous sector studies the equilibrium structure of the economic geography. As reality is seldom in a stationary state, it is important to see the motion of the system over time and space. As we have already provided the procedure to follow the motion of each variable in the system, it is straightforward to plot the motion with a computer. First, let us examine the motion of the system when the initial condition is away from the stationary state with $K_i(0) = 100$. The initial state of the capital stocks in the housing sector is lower than its corresponding long-run equilibrium value for the given technology and preference. As shown in Figure 7.2.5, as time passes, the national wealth, output, the capital stocks employed by the two sectors, the wage rate and consumption are increased, and the rate of interest is reduced. At any given location, the housing rent, land rent and housing consumption rise in time.

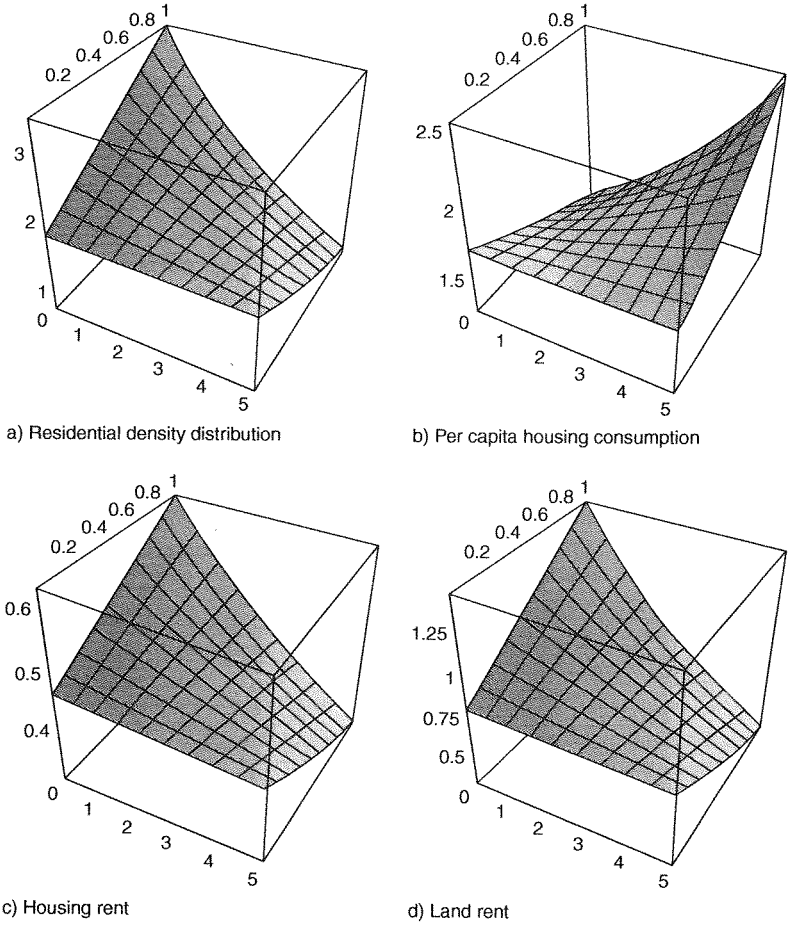


Figure 7.2.3 The economic geography and the propensity to use leisure.

We now study possible effects of changes in the preference. As we require $\xi + \eta + \lambda = 1$ holds at any point of time, any change in the three propensities has to satisfy this requirement. We now consider a case that the propensity to save falls and the propensity to consume housing rises as follows:

$$\Delta\eta = 0.02,\ \Delta\lambda = -0.02.$$

We illustrate the change rates as in Figure 7.2.6. In the plots, a variable $\overline{\Delta}x(t)$ stand for the change rate of the variable $x(t)$ in percentage due to changes in the parameter value. That is,

$$\overline{\Delta}x(t) \equiv \frac{x(t;A_1) - x(t;A_{10})}{x(t;A_{10})} \times 100,$$

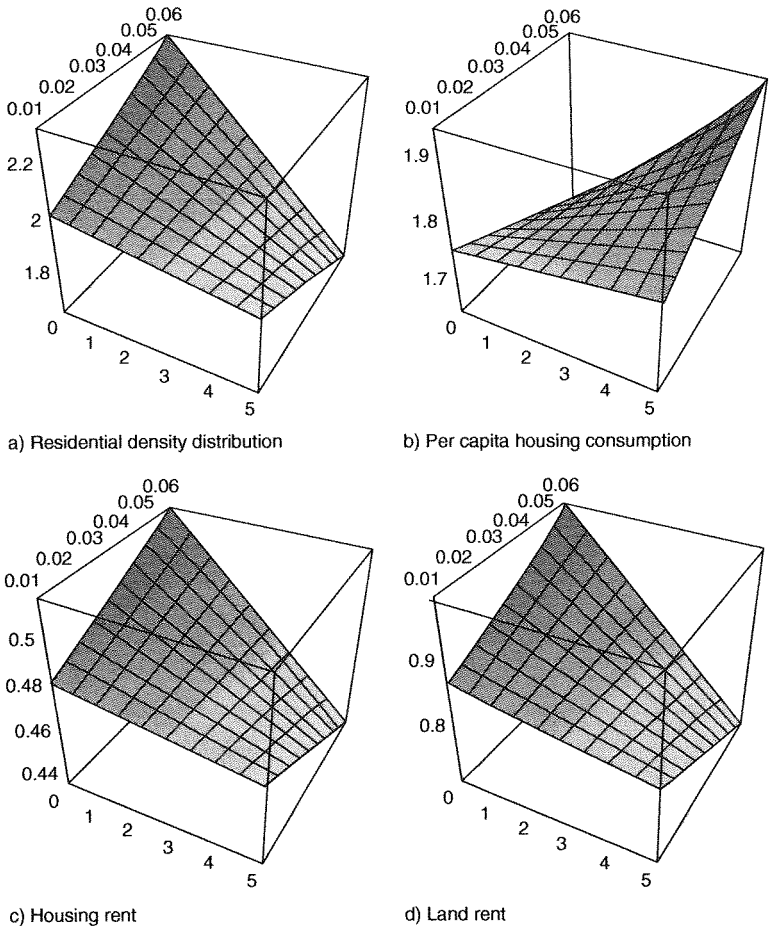


Figure 7.2.4 The economic geography and the travel speed.

where $x(t; A_1)$ stands for the value of the variable, x, with the parameter value A_1 at time t, and $x(t; A_{10})$ stands for the value of the variable, x, with the parameter value A_{10} at time t.

From Figure 7.2.6, as the propensities are changed, the national wealth and the output level of the industrial sector rise initially and then fall down. The capital stocks employed by the housing sector rises and the capital stocks employed by the industrial sector falls. The wage rate falls. The disposable income and consumption level rise initially and then fall. The rate of interest is increased. The housing rent, the land rent, and housing consumption are increased. The land rent increases more than the housing rent because the propensity for housing is increased and the rate of interest rises.

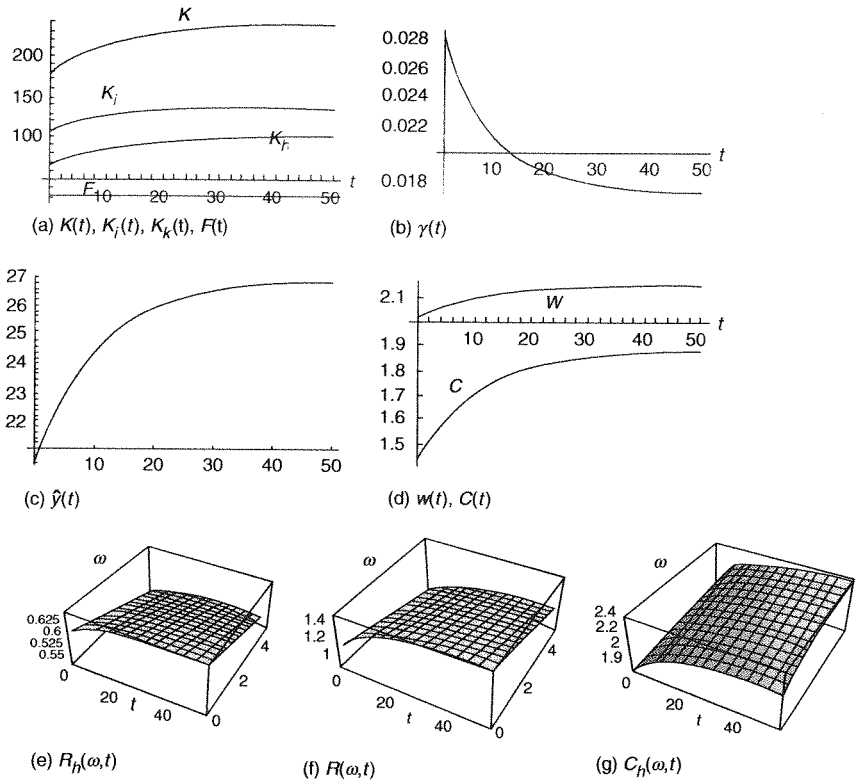


Figure 7.2.5 The motion of the system.

7.3 The dynamics of the monetary economy

We are now concerned with the case of $\varepsilon_0 > 0$.

7.3.1 The motion of the monetary system and steady state

Section 7.2 examines the dynamics when $\varepsilon_0 = 0$. We now examine the dynamics of the monetary economy when $\varepsilon_0 > 0$. We omit ω or/and t in the expressions, wherever without causing confusion.

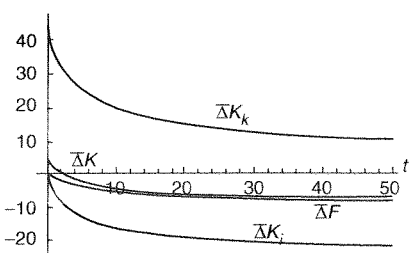
Lemma 7.3.1

The dynamics of the national wealth and real money balances are determined by

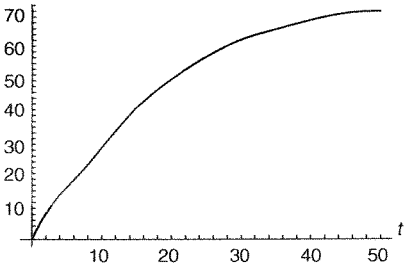
$$\dot{K} = \Omega_K(K, m),$$

$$\dot{m} = \Omega_m(K, m),$$

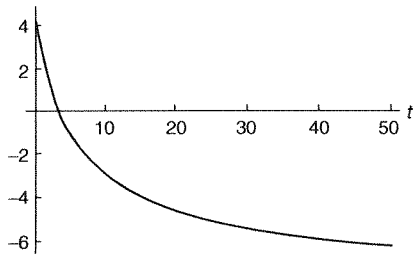
in which $\Omega_K(K, m)$ and $\Omega_m(K, m)$ are functions of K and m explicitly defined in Appendix 7.2. At any point of time, all the other variables are determined as unique functions of K and m by the following procedure: $\overline{m} = m \to \pi = \Omega_\pi(K, m) \to K_i$



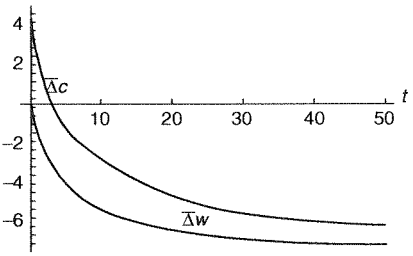
(a) $\overline{\Delta}K(t)$, $\overline{\Delta}K_i(t)$, $\overline{\Delta}K_k(t)$, $\overline{\Delta}F(t)$



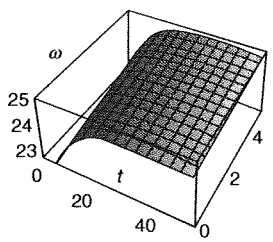
(b) $\overline{\Delta}y(t)$



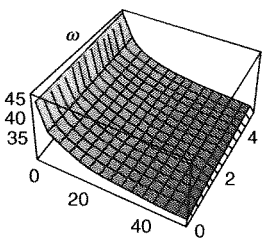
(c) $\overline{\Delta}\hat{y}(t)$



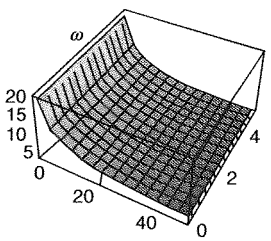
(d) $\overline{\Delta}w(t)$, $\overline{\Delta}c(t)$



(e) $\overline{\Delta}R_k(\omega, t)$



(f) $\overline{\Delta}R(\omega, t)$



(g) $\overline{\Delta}c_k(\omega, t)$

Figure 7.2.6 Changes in the propensities to consume housing and to save.

by (A.7.2.3) $\rightarrow K_h = K - K_i \rightarrow F(K_i, N) \rightarrow r$ and w by (7.1.1) $\rightarrow \overline{Y} \rightarrow C$ and S by (A.7.2.1) $\rightarrow \overline{r}$ by (A.7.2.8) $\rightarrow k = K/N \rightarrow \overline{y} = (1+r)k + w + \mu m + \overline{r} \rightarrow k_h(\omega)$ by (A.7.2.14) $\rightarrow n(\omega)$ by (A.7.2.16) and (A.7.2.17) $\rightarrow R_h(\omega)$ by (A.7.2.15) $\rightarrow L_h(\omega) = 1/n(\omega) \rightarrow c_h(\omega) = k_h^{\alpha_h}(\omega) L_h^{\beta_h}(\omega) \rightarrow R(\omega)$ by (7.1.3) $\rightarrow c$ and s by (7.1.10).

We now find out steady states, using the results that $\overline{y}$, k, a, s, $\overline{m}$, and c are invariant over space. By equations (7.1.11) and (A.7.2.11) at steady state we have

$$\left(\mu - \delta_k + F_{K_i}\right) m - \frac{\varepsilon \overline{Y}}{N} = 0,$$

$$s = a. \tag{7.3.1}$$

From equation (A.7.2.3) we solve

$$K = \frac{\overline{\alpha} F_{K_i} K_i + F + \left(\delta + \mu + F_{K_i}\right) mN}{\overline{\alpha} F_{K_i} - \delta}. \tag{7.3.2}$$

Insert equation (7.3.2) in equation (A.7.2.5)

$$\overline{Y} = \frac{F + \delta K_i + \left(\delta + \mu + F_{K_i}\right) Nm}{\alpha_h \eta \left(\overline{\alpha} F_{K_i} - \delta\right)} F_{K_i}. \tag{7.3.3}$$

For $\overline{Y} > 0$, we should require $F_{K_i} > \delta / \overline{\alpha}$. From this equation and the first equation in (7.3.1), we solve

$$m = \frac{(F + \delta K_i) \varepsilon F_{K_i}}{\left(\overline{\alpha} \alpha_h \eta F_{K_i} - \varepsilon F_{K_i} - \alpha_h \eta \delta\right)\left(\mu - \delta_k + F_{K_i}\right) N}. \tag{7.3.4}$$

Assume that $\delta / \overline{\alpha} > \mu - \delta_k$. This requirement is not strict as $\mu - \delta_k$ tends to be small. Under this requirement, for $m > 0$ we should require $(\overline{\alpha} \alpha_h \eta - \varepsilon) F_{K_i} > \alpha_h \eta \delta$, that is,

$$F_{K_i} > \frac{\alpha_h \eta \delta}{\xi + \lambda}. \tag{7.3.5}$$

If this inequality is satisfied, then $F_{K_i} > \delta / \overline{\alpha}$ is guaranteed.

From $s = a$, $s = \lambda \overline{y}$, and $a = k + m$, we have

$$\lambda \overline{Y} = K + mN.$$

Substituting equations (7.3.2)–(7.3.4) into the above equation yields

$$\Omega(K_i) = \left[\lambda F - \frac{\alpha_h \eta F}{F_{K_i}} - \overline{\xi} K_i\right] \frac{\left((\xi + \lambda) F_{K_i} - \alpha_h \eta \delta\right)\left(\overline{\mu} + F_{K_i}\right)}{(F + \delta K_i) \varepsilon}$$
$$+ \left(\overline{\lambda} + \lambda F_{K_i}\right) F_{K_i} - \alpha_h \eta \mu = 0, \tag{7.3.6}$$

where

$$\overline{\xi} \equiv \xi + \varepsilon + \delta_k \lambda > 0, \quad \overline{\mu} \equiv \mu - \delta_k, \quad \overline{\lambda} \equiv \overline{\mu} \lambda - \alpha_h \eta - \xi - \varepsilon.$$

Let $\overline{K}_i$ be the value of K_i at which $F_{K_i} = \alpha_h \eta \delta / (\xi + \lambda)$. As F_{K_i} falls in K_i, we should require $0 < K_i < \overline{K}_i$. From $F_{K_i} \to +\infty$ as $K_i \to 0$, we see that for sufficiently small K_i, we should have $\Omega(K_i) > 0$ and

$$\Omega\left(\overline{K}_i\right) = \left(\overline{\mu} \lambda - \xi - \varepsilon - \left(1 - \frac{\delta \lambda}{(\xi + \lambda)}\right) \alpha_h \eta\right) \frac{\alpha_h \eta \delta}{(\xi + \lambda)} - \alpha_h \eta \mu < 0,$$

where we assume that $\overline{\mu}$ is sufficiently small when it is positive. We thus conclude that equation (7.3.6) has at least one solution, satisfying $0 < K_i < \overline{K}_i$. From equation (7.3.6), we have

$$\frac{d\Omega}{dK_i} = \left(\lambda F_{K_i} + \frac{\alpha_h \eta F F_{K_i K_i}}{F_{K_i}^2} - \alpha_h \eta - \overline{\xi}\right) \frac{(\xi_0 F_{K_i} - \alpha_h \eta \delta)(\overline{\mu} + F_{K_i})}{(F + \delta K_i)\varepsilon}$$
$$+ \left(\overline{\lambda} + 2\lambda F_{K_i}\right) F_{K_i K_i} + \frac{1}{(F + \delta K_i)\varepsilon} \left(\lambda F - \frac{\alpha_h \eta F}{F_{K_i}} - \overline{\xi} K_i\right)$$
$$\left\{(\overline{\mu} + F_{K_i})\xi_0 F_{K_i K_i} + (\xi_0 F_{K_i} - \alpha_h \eta \delta)\left[F_{K_i K_i} - \left(\frac{F_{K_i} + \delta}{F + \delta K_i}\right)(\overline{\mu} + F_{K_i})\right]\right\},$$

We cannot explicitly determine the sign of $d\Omega/dK_i$. If we have $d\Omega/dK_i > 0$, the problem has a unique steady state. Otherwise, it may have multiple steady states.

Lemma 7.3.2

The dynamic system has at least one steady state. The variables are determined by the following procedure: K_i by (7.3.6) $\to m$ by (7.3.4) $\to \overline{m} = m \to \pi = \mu \to K$ by (7.3.2) $\to K_h = K - K_i \to F(K_i, N) \to r$ and w by (7.1.1) $\to \overline{Y} = (K + mN)/\lambda \to C = \xi \overline{Y}$ and $S = \lambda \overline{Y} \to \overline{r}$ by (A.7.2.8) $\to k = K/N \to \overline{y} = \overline{Y}/N \to k_h(\omega)$ by (A.7.2.14) $\to n(\omega)$ by (A.7.2.16) and (A.7.2.17) $\to R_h(\omega)$ by (A.7.2.15) $\to L_h(\omega) = 1/n(\omega) \to c_h(\omega) = k_h^{\alpha_h}(\omega) L_h^{\beta_h}(\omega) \to R(\omega)$ by (7.1.3) $\to c$ and s by (7.1.10).

Although we can provide stability conditions, we omit the issue as the expressions are tedious and the economic implications are difficult to explicitly interpret.

7.3.2 The impact of the inflation rate

We now examine effects of change on the equilibrium. Taking derivatives of equation (7.3.6) with respect to μ yields

$$\frac{d\Omega}{dK_i}\frac{dK_i}{d\mu} = \frac{\left(\xi + \varepsilon - (\lambda - \mu)(\overline{\mu} + F_{K_i})\right) F_{K_i} + \delta_k \alpha_h \eta}{(\overline{\mu} + F_{K_i})}. \tag{7.3.7}$$

As the term, $d\Omega/dK_i$, is ambiguous and it is reasonable to assume that $\lambda - \mu$ is positive (which implies that the right-hand side of the equation is ambiguous), the sign of $dK_i/d\mu$ is ambiguous. Nevertheless, if $\overline{\mu} + F_{K_i}$ is large, the right-hand side is negative. If $d\Omega/dK_i$ is negative, then $dK_i/d\mu$ is positive. From equations (7.1.1),

if K_i rises (falls), then the rate of interest falls (rises) and the national output level, the per-capita output level and the wage rate rise (fall).

From equation (7.3.4), we have

$$\frac{dm}{d\mu} = \left[\frac{F_{K_i} + \delta}{F + \delta K_i} + \frac{F_{K_i K_i}}{F_{K_i}} + \frac{F_{K_i K_i}}{\overline{\mu} + F_{K_i}} - \frac{(\overline{\alpha}\alpha_h \eta - \varepsilon) F_{K_i K_i}}{\overline{\alpha}\alpha_h \eta F_{K_i} - \varepsilon F_{K_i} - \alpha_h \eta \delta} \right] \frac{dK_i}{d\mu} - \frac{1}{\overline{\mu} + F_{K_i}}. \tag{7.3.8}$$

As $F_{K_i K_i} < 0$, the right-hand side of the equation may be either positive or negative, even when $dK_i/d\mu$ is positive. From $\overline{m} = m$, we get the impact on $\overline{m}$. From equations (7.3.1), we have

$$\frac{1}{\overline{Y}} \frac{d\overline{Y}}{d\mu} = \frac{1}{C} \frac{dC}{d\mu} = \frac{1}{S} \frac{dS}{d\mu} = \frac{1}{m} \frac{dm}{d\mu} - \frac{F_{K_i K_i}}{\overline{\mu} + F_{K_i}} \frac{dK_i}{d\mu}.$$

We also have

$$\frac{1}{\overline{y}} \frac{d\overline{y}}{d\mu} = \frac{1}{c} \frac{dc}{d\mu} = \frac{1}{s} \frac{ds}{d\mu} = \frac{1}{\overline{Y}} \frac{d\overline{Y}}{d\mu}.$$

From equations (A.7.2.15)–(A.7.2.17), we have

$$\frac{dn(\omega)}{d\mu} = 0,$$

$$\frac{1}{R(\omega)} \frac{dR(\omega)}{d\mu} = \frac{\alpha_h F_{K_i K_i}}{r + \delta_k} \frac{dK_i}{d\mu} + \frac{\beta_h}{\overline{y}} \frac{d\overline{y}}{d\mu},$$

$$\frac{1}{k_h} \frac{dk_h}{d\mu} = -\frac{F_{K_i K_i}}{r + \delta_k} \frac{dK_i}{d\mu} + \frac{1}{\overline{y}} \frac{d\overline{y}}{d\mu}.$$

From equation (A.7.2.7), we have

$$\frac{1}{\overline{r}} \frac{d\overline{r}}{d\mu} = \frac{F_{K_i K_i}}{r + \delta_k} \frac{dK_i}{d\mu} + \frac{1}{\overline{y}} \frac{d\overline{y}}{d\mu}$$

It is straightforward to express the effects of the change in μ on all the other variables. We will not represent the analytical results.

7.4 Concluding remarks

This chapter proposed a spatial monetary growth model by synthesizing the main ideas in the three key models in the neoclassical growth theory and urban economics. The non-monetary economic growth with economic geography has

a unique long-run stable equilibrium when the production function takes on the Cobb-Douglas form. We also analyzed changes in the parameters upon the non-monetary system. We have limited our study to an extremely simplified spatial structure of the economic system. Many limitations of this model become apparent in the light of the sophistication of the literature of regional science and urban economics with regard to economic structure. Nevertheless, this chapter has its merit in that it synthesizes three key models in the growth theory and urban economics in a compact framework. Numerous directions exist for extending the model. The Solow model is the key model in the neoclassical economic growth theory and Alonso model is the key model in the modern urban economics. Numerous meaningful extensions of either of the two models have already been carried out.[23] We may introduce more realistic representations of housing market dynamics and transportation systems with congestion. We now point out a few more straightforward extensions. For instance, we may consider the economy as a small country. We may consider a small economy has negligible impact on the interest rate in a globally open market. This assumption has been accepted in the literature of international economics. There are many small-country growth models in the literature. With r being fixed, we can examine the residential distribution, land rent distribution, and economic growth by our model. It can be seen that the mathematical analysis becomes easier for an open small economy. Another extension is to assume that the economy is a small and open city in the national economy. In this case, both the utility level and the rate of interest should be exogenously fixed at the national levels. To explain urban configuration for this linear city, we may take account of communications among firms as by Borokov and Hochman (1977) and O'Hara (1977). For instance, we may consider the CBD as a linear area. Firms are assumed to be identical. Firms use floor space and interact among themselves and they have to bear transaction costs. Transaction cost functions may take on the form as suggested by Beckmann for social interactions

$$T(\omega_2) = \int_{-L_u}^{L_u} \tau \left|\omega_1 - \omega_2\right| m(\omega_1) d\omega_1,$$

where L_u is the boundary between the residential area and the CBD and T is the cost for the firm located at ω_2 in the CBD. Within the framework proposed in this study, we can explain, at least in principle, spatial configuration of the CBD. In the literature of the modern urban economics, CBD is no more fixed as assumed in this study. As shown in Fujita and Thisse (2002b: Chapter 6) by combining the basic ideas by Beckmann and Borokov and Hochman mentioned above, various urban forms can be explained due to communication externalities.[24] Another extension is in the area of housing taxation.[25] It would be interesting to examine how the presence of exogenous growth mechanisms would change the analysis of the tax incidence of housing assets or the effects of different taxation of different assets.

APPENDIX

A.7.1 Proving lemma 7.2.1

First, we define the total disposal income as

$$\hat{Y}(t) \equiv \int_0^L \hat{y}(\omega, t) n(\omega, t) d\omega.$$

We now show that this variable can be expressed as a unique function of the total wealth, $K(t)$. Multiplying all the equations in (7.1.10) by $n(\omega, t)$ and then integrating the resulted equations from 0 to L with respect to ω, we obtain

$$C = \xi \hat{Y}, (r + \delta_k) \frac{K_h}{\alpha_h} = \eta \hat{Y}, \quad S = \lambda \hat{Y}, \tag{A.7.1.1}$$

where we use $R_h c_h = (r + \delta_k) k_h / \alpha_h$ from equations (7.1.3).

The national production is equal to the national consumption and national net saving. That is[26]

$$C(t) + S(t) - K(t) + \delta_k K(t) = F(t),$$

where $S(t)$ is the total saving of the economy

$$S(t) = \int_0^L s(\omega, t) n(\omega, t) d\omega.$$

Substituting S and C in equations (A.7.1.1) into the above equation yields

$$(\xi + \lambda) \hat{Y} + \delta_k K = F + K. \tag{A.7.1.2}$$

From $r + \delta_k = F_{K_i}$ in equations (7.1.1) and $(r + \delta_k) K_h = \alpha_h \eta \hat{Y}$ in equations (A.7.1.1), we have

$$F_{K_i} K_h = \alpha_h \eta \hat{Y}.$$

Substituting equation (A.7.1.2) into the above equation yields

$$\Lambda(K_i) \equiv \left(F_{K_i} - \delta \overline{\alpha}\right) K - F_{K_i} K_i - \overline{\alpha} F = 0, \tag{A.7.1.3}$$

where we use

$$K_h = K - K_i, \quad \delta \equiv 1 - \delta_k, \quad \overline{\alpha} \equiv \frac{\alpha_h \eta}{\xi + \lambda} > 0.$$

We now show that for any given $K > 0$ which satisfies $F(K) > \delta_k K$, then equation (A.7.1.3), $\Lambda(K_i) = 0$, has a unique solution for $0 < K_i < K$. As

$$\Lambda(0) = \left(F_{K_i}(0) - \delta\overline{\alpha}\right)K > 0,$$

$$\Lambda(K) = -\delta\overline{\alpha}K - \overline{\alpha}F < 0,$$

$$\frac{d\Lambda}{dK_i} = F_{K_iK_i}(K - K_i) - F_{K_i}(1 + \overline{\alpha}) < 0, \quad K > K_i > 0, \tag{A.7.1.4}$$

we can consider K_i as a function of K. We denote this unique relation by $K_i = \overline{\Lambda}(K)$. We have

$$K_i = \overline{\Lambda}(0) = 0, \quad K_i = \overline{\Lambda}(+\infty) = +\infty.$$

The properties are obvious from equation (A.7.1.3) if we note that K_i and F_{K_i} are non-negative. Taking the derivatives of equation (A.7.1.3) with respect to K yields

$$\frac{d\overline{\Lambda}}{dK} = -\left(F_{K_i} - \delta\overline{\alpha}\right)\left(\frac{d\Lambda}{dK_i}\right)^{-1} > 0, \tag{A.7.1.5}$$

where $\left(F_{K_i} - \delta\overline{\alpha}\right) > 0$ and $(d\Lambda/dK_i)^{-1} < 0$. That is, $\overline{\Lambda}' > 0$. An increase in the total capital brings about an increase in the capital employed by the industrial sector.

From $K_i = \overline{\Lambda}(K)$ and $K_h = K - K_i$, we see that the capital stocks, K_i and K_h, employed by the industrial and housing sectors are uniquely determined as functions of the total capital stocks, K, at any point of time. From $K_h = K - K_i$, and equations (A.7.1.4) and (A.7.1.5), we have the impact of changes in K on K_h as follows

$$\frac{dK_h}{dK} = \frac{F_{K_iK_i}(K - K_i) - \overline{\alpha}F_{K_i} - \delta\overline{\alpha}}{F_{K_iK_i}(K - K_i) - F_{K_i}(1 + \overline{\alpha})} > 0.$$

An increase in the total capital stock increases the capital stocks employed by the housing sector.

From $F = F\left(\overline{\lambda}, N\right)$, F is also a unique function of K. Solve equation (A.7.1.2) in $\hat{Y}$

$$\hat{Y} = \tilde{\Lambda}(K) \equiv \frac{F + \delta K}{\xi + \lambda}. \tag{A.7.1.6}$$

At any point of time, the dynamic equation, (A.7.1.18), determines the value of the total capital stocks, $K(t)$. We can show that all the other variables are uniquely determined as a function of K and ω $(0 \leq \omega \leq L)$ at any point of time. From equations (A.7.1.6) and (A.7.1.3), we get $\hat{Y}$ and K_i. We have $K = K - K_i$. From equations (A.7.1.1) and (7.1.1), we get C, r, S, and w.

We now show that the residential density distribution is invariant in time. That is, $dn/dt = 0$ for the given technologies, preference and transportation system. By equations (7.1.3), we have

$$R(t) = \frac{(r(K(t)) + \delta_k)\beta_h k_h n(\omega, t)}{\alpha_h}, \quad 0 \le \omega \le L.$$

Integrate the above equation from 0 to L

$$\bar{r} = \frac{(r + \delta_k)\beta_h K_h}{\alpha_h N}. \tag{A.7.1.7}$$

As r and K_h are functions of K, we see that the per-capital income from land is also a function of $K(t)$ at any point of time.

By equations (7.1.6) and (7.1.7) and the definition of $\hat{y}(\omega, t)$, we have

$$\hat{y}(\omega) = (1 + r)k(\omega) + w + \bar{r}. \tag{A.7.1.8}$$

Insert this equation in equation (7.1.11)

$$\dot{k}(\omega, t) = (\lambda - 1 + \lambda r(K(t)))k(\omega, t) + \lambda[w(K(t)) + \bar{r}(K(t))], \quad 0 \le \omega \le L, \tag{A.7.1.9}$$

where we use $s = \lambda\hat{y}$. This is a linear differential equation in $k(\omega, t)$. For convenience of expression, we rewrite equation (A.7.1.9) as

$$\dot{k}(\omega, t) + h_1(t)k(\omega, t) = h_2(t), \quad 0 \le \omega \le L,$$

where

$$h_1(t) = 1 - \lambda - \lambda r(K(t)), h_2(t) \equiv \lambda[w(K(t)) + \bar{r}(K(t))], \quad 0 \le \omega \le L.$$

Applying the general solution for this type of linear differential equations, we get

$$k(\omega, t) = e^{-\int h_1(\tau)d\tau}\left(h_0 + \int h_2(\tau)e^{\int h_1(\tau)d\tau}d\tau\right), \tag{A.7.1.10}$$

where h_0 is a constant to be determined by initial conditions. Equation (A.7.1.10) tells that the path of households' wealth over time and space is given as a function of the total wealth of the economy. From equation (A.7.1.10), as h_1 and h_2 are independent of location, we see that if the initial distribution of the wealth is not dependent on location,[27] h_0 is also independent of the location. From equation (A.7.1.10), we conclude that $k(\omega, t)$ is independent of location, that is,

$$k(\omega_1) = k(\omega_2), \quad 0 \le \omega_1, \quad \omega_2 \le L.$$

In this study, we assume that the initial wealth distribution is independent of location, ω.We thus have

$$k(t) = \frac{K(t)}{N}.$$

By this property and equations (A.7.1.8), we have

$$\hat{y}(\omega_1) = \hat{y}(\omega_2), \quad 0 \leq \omega_1, \quad \omega_2 \leq L.$$

As shown later on, this property greatly simplifies our analysis. This is the technical reason that we omit transportation costs in this model. If we introduce transportation costs as a function of the distance to the CBD, then the distance variable ω, will appear in h_2. It can be seen that the wealth distribution is dependent on ω. For the analytical tractability, we omit the transportation costs. We take account of distance in the residential location through the time distribution between leisure and travel.

It should be remarked that by equation (A.7.1.10), we can find an equation to determine $K(t)$. Multiplying the two sides of equation by $n(\omega, t)$ and then integrating from 0 to L with regard to the distance, we have

$$K(t) = e^{-\int_0^t h_1(\tau)d\tau} N \left(h_0 + \int_0^t h_2(\tau) e^{\int h_1(\tau)d\tau} d\tau \right).$$

We will find a differential equation to determine $K(t)$.

We have shown that the per-capita wealth and disposal income are invariant in space but changeable over time. We now show that the residential density distribution is invariant in time but changeable over time.

Substituting equations (7.1.10) into $U(\omega, t)$ in (7.1.9) and then using $U(0) = U(\omega)$, we have

$$\frac{R_h(\omega)}{R_h(0)} = \left(\frac{n(\omega)}{n(0)} \right)^{\vartheta/\eta} \left(\frac{T_h(\omega)}{T_h(0)} \right)^{\sigma/\eta}, \tag{A.7.1.11}$$

where we use $\hat{y}(0) = \hat{y}(\omega)$. From $R_h(\omega)c_h(\omega) = \eta\hat{y}(\omega)$ and $(r + \delta_k)k_h = \alpha_h R_h c_h$, we have

$$k_h(\omega) = \frac{\alpha_h \eta \hat{y}}{r + \delta_k}. \tag{A.7.1.12}$$

Substituting $c_h = k_h^{\alpha_h} L_h^{\beta_h}$ into $(r + \delta_k)k_h = \alpha_h c_h R_h$, we have

$$R_h(\omega) = \eta^{\beta_h} \left(\frac{r + \delta_k}{\alpha_h} \right)^{\alpha_h} n^{\beta_h}(\omega) \hat{y}^{\beta_h}. \tag{A.7.1.13}$$

Substitute equation (A.7.1.13) into equation (A.7.1.11)

$$n(\omega) = n(0)\left(\frac{T_h(\omega)}{T_h(0)}\right)^{\beta_0}, \tag{A.7.1.14}$$

where

$$\beta_0 \equiv \frac{\sigma}{\eta\beta_n - \vartheta}.$$

Integrating equation (A.7.1.14), we obtain

$$n(0,t) = \frac{NT_h^{\beta_0}(0)}{\int_0^L T_h^{\beta_0}(\omega)d\omega}. \tag{A.7.1.15}$$

As $T_h(\omega)$ is explicitly defined as a function of location and independent of time, we see that $n(0,t)$ is independent of time. By equation (A.7.1.14), we conclude that $n(\omega,t)$ is actually independent of time but dependent on location. This property is important for us to find a differential equation for $K(t)$.

We now determine the differential equation for $K(t)$. Substitute $s = \lambda\hat{y}$ from equations (7.1.10) into equation (7.1.11)

$$\dot{k} = \lambda\hat{y} - k, \quad 0 \le \omega \le L. \tag{A.7.1.16}$$

Multiplying (A.7.1.16) by $n(\omega,t)$ and then integrating the equation result from 0 to L with respect to ω yields

$$\dot{K} = \lambda\hat{Y} - K, \tag{A.7.1.17}$$

where we use

$$\dot{K}(t) = \int_0^L \frac{d(k(\omega,t)n(\omega,t))}{dt}d\omega = \int_0^L \dot{k}(\omega,t)n(\omega,t)d\omega,$$

in which we use $\dot{n} = 0$.

Substitute equation (A.7.1.6) into equation (A.7.1.17)

$$\dot{K} = \frac{\lambda F - (\xi + \lambda\delta_k)K}{\xi + \lambda}. \tag{A.7.1.18}$$

This one-dimensional differential equation contains a single variable, K. Instead of this equation, for convenience of analysis we now express the dynamics in terms

of $K_i(t)$. From equation (A.7.1.5), we solve $K_i(t)$ as a unique function of $K(t)$ as follows

$$K = \frac{F_{K_i}K_i + \overline{\alpha}F}{F_{K_i} - \delta\overline{\alpha}}, \tag{A.7.1.19}$$

where $F_{K_i} - \delta\overline{\alpha} > 0$. Taking derivatives of equation (A.7.1.19) with respect to time yields

$$\dot{K} = \Lambda^*(K_i)\dot{K}_i, \tag{A.7.1.20}$$

in which

$$\Lambda^*(K_i) \equiv \frac{\left(F_{K_i} - \delta\overline{\alpha}\right)(1+\overline{\alpha})F_{K_i} - (\delta K_i + F)\overline{\alpha}F_{K_iK_i}}{\left(F_{K_i} - \delta\overline{\alpha}\right)^2} > 0.$$

As $F_{K_iK_i} < 0$ and $F_{K_i} - \delta\overline{\alpha} > 0$, we see that $\Lambda^* > 0$. Inserting equations (A.7.1.19) and (A.7.1.20) in equation (A.7.1.18), we obtain equation (7.2.1), which determines the motion of $K_i(t)$.

Under $F_{K_i} > \delta\overline{\alpha}$, an equilibrium point of the dynamic system is given by

$$\lambda F_{K_i}F - (\xi + \lambda\delta_k)F_{K_i}K_i - (\xi + \lambda)\overline{\alpha}F = 0.$$

Denote K_i^* the value of K_i which satisfies $F_{K_i} = \delta\overline{\alpha}$. According to the properties of F, we see that the solution exists and is unique. As F_{K_i} declines in K_i, we should search equilibrium for $0 < K_i < K_i^*$. Hence, the equilibrium problem is to solve the following equation

$$\Omega(K_i) \equiv \left(\lambda - (\xi + \lambda\delta_k)\frac{K_i}{F}\right)F_{K_i} - (\xi + \lambda)\overline{\alpha} = 0, \quad 0 < K_i < K_i^*. \tag{A.7.1.21}$$

As $K_i/F \to 0$ and $F_{K_i} \to \infty$ as $K_i \to 0$, we have $\Omega(0) > 0$. We also have

$$\Omega(K_i^*) = -(\xi + \lambda\delta_k)\delta\overline{\alpha}\frac{K_i^*}{F} - (\xi + \delta_k\lambda)\overline{\alpha} < 0.$$

We thus conclude that the equation, $\Omega(K_i) = 0$, has at least one solution for $0 < K_i < K_i^*$. Take derivatives of $\Omega(K_i)$ with respect to K_i

$$\Omega' = \left(\frac{F}{K_i} - \frac{\xi}{\lambda} - \delta_k\right)\frac{\lambda K_i F_{K_iK_i}}{F} - \left(\frac{F}{K_i} - F_{K_i}\right)\left(\frac{\xi + \lambda\delta_k}{F^2}\right)K_iF_{K_i}.$$

If Ω' is negative for $0 < K_i < K_i^*$, then $\Omega(K_i) = 0$ has a unique solution. As $F/K_i - F_{K_i} > 0$ and

$$\left(\frac{F}{K_i}\right)' = \frac{F' - F/K_i}{K_i} < 0,$$

we see that if $F/K_i > \xi/\lambda + \delta_k$ at $K_i = K_i^*$, then Ω' is negative for $0 < K_i < K_i^*$. It should be remarked that even if $F/K_i < \xi/\lambda + \delta_k$ at $K_i = K_i^*$, we might still have $\Omega' < 0$ for $0 < K_i < K_i^*$.

In summary, we have guaranteed at least one equilibrium point. An equilibrium point is stable (unstable) if the number

$$\left(\frac{F}{K_i}-\frac{\xi}{\lambda}-\delta_k\right)\lambda F_{K_iK_i}K_i+\left(\frac{F}{K_i}-F_{K_i}\right)\lambda\left(\overline{\alpha}\left(\frac{\xi}{\lambda}+1\right)-F_{K_i}\right),$$

which is evaluated at the equilibrium point, is negative (positive). As the sign of the number is ambiguous, we cannot explicitly judge the stability. We have thus proved Lemma 1.

A.7.2 Proving lemma 7.3.1

First, we define the total disposal income as

$$\overline{Y}(t)\equiv\int_0^L \bar{y}(\omega,t)n(\omega,t)d\omega.$$

We now show that this variable can be expressed as a unique function of $K(t)$ and $m(t)$.

Multiplying all the equations in (7.1.10) by $n(\omega,t)$ and then integrating the resulted equations from 0 to L with respect to ω, we obtain

$$mN=\frac{\varepsilon\overline{Y}}{r+\pi},C=\xi\overline{Y},(r+\delta_k)\frac{K_h}{\alpha_h}=\eta\overline{Y},S=\lambda\overline{Y}, \tag{A.7.2.1}$$

where we use $R_hc_h=(r+\delta_k)k_h/\alpha_h$ from equations (7.1.3).

It should be noted that the national production is equal to the national consumption and national net saving. That is

$$C+S-K-mN-(\mu-\pi)mN+\delta_kK=F.$$

Substituting equations (A.7.2.1) into the above equation yields

$$(\xi+\lambda+\varepsilon)\overline{Y}=F+\delta K+\left(\delta+\mu+F_{K_i}\right)mN, \tag{A.7.2.2}$$

where we use $r+\delta_k=F_{K_i}$ in (7.1.1) and $\delta\equiv 1-\delta_k$. From $r+\delta_k=F_{K_i}$ and $(r+\delta_k)K_h=\alpha_h\eta\overline{Y}$ in (A.7.1.1), we have

$$F_{K_i}K_h=\alpha_h\eta\overline{Y}.$$

Substituting equation (A.7.2.2) into the above equation yields

$$\Lambda(K_i)\equiv\left(\overline{\alpha}F_{K_i}-\delta\right)K-\overline{\alpha}F_{K_i}K_i-F-\left(\delta+\mu+F_{K_i}\right)mN, \tag{A.7.2.3}$$

where we use

$$K_h = K - K_i,\ \overline{\alpha} \equiv \frac{\xi + \lambda + \varepsilon}{\alpha_h \eta} > 0.$$

We now examine under what conditions equation, $\Lambda(K_i) = 0$, has solution fo $0 < K_i < K$. As

$$\Lambda(0) = \left[F_{K_i} - \frac{\delta}{\overline{\alpha}} - \left(\delta + \mu + F_{K_i}\right) \frac{mN}{\overline{\alpha} K} \right] \overline{\alpha} K,$$

$$\Lambda(K) = -\delta K - F - \left(\delta + \mu + F_{K_i}\right) mN < 0.$$

$$\Lambda'(K_i) = \overline{\alpha} F_{K_i K_i} \left(K - \frac{mN}{\overline{\alpha}} - K_i \right) - \overline{\alpha} F_{K_i} - F_{K_i}, \qquad \text{(A.7.2.4}$$

where we require $\lim F_{K_i} K_i \to 0$ as $K_i \to +0$. As $F_{K_i K_i} < 0$ and $F_{K_i} \to +\infty$, w see that if $m < \overline{\alpha} K / N$, then we have $\Lambda(0) > 0$ and $\Lambda'(K_i) < 0$. The equation ha a unique solution, satisfying $0 < K_i < K$. From the definition of $\overline{\alpha}$, this paramete should be larger than unit. As K/N is the per capita physical wealth and m is the pe capita monetary wealth, the requirement, $m < \overline{\alpha} K / N$, is reasonable. We denote th unique solution by $K_i = \overline{\Lambda}(K, m)$. From $K_i = \overline{\Lambda}(K, m)$ and $K_h = K - K_i$, we se that the capital stocks, K_i and K_h, employed by the industrial and housing sector are uniquely determined as functions of the total capital stocks, K, at any point o time.

From $F = F\left(\overline{\lambda}, N\right)$, F is also a unique function of $K$and m. Solve equatio (A.7.2.2) in $\overline{Y}$

$$\overline{Y} = \tilde{\Lambda}(K, m) \equiv \frac{F + \delta K + \left(\delta + \mu + F_{K_i}\right) mN}{\xi + \lambda + \varepsilon}. \qquad \text{(A.7.2.5}$$

From $mN = \varepsilon \overline{Y} / (r + \pi)$ and $\dot{m} = (\mu - \pi) m$, we have

$$\dot{m} = \Omega_m(K, m) \equiv \left(\mu - \delta_k + F_{K_i}\right) m - \frac{\varepsilon \tilde{\Lambda}(K, m)}{N}, \qquad \text{(A.7.2.6}$$

where we use $r + \delta_k = F_{K_i}$. We now show that all the other variables are uniquel determined as functions of $K(t)$ and $m(t)$ at any location and any point of time. I is straightforward to see that $\overline{Y}$, K_i, $K = K - K_i$, π, C, r, S, and w are function of $K(t)$ and $m(t)$.

We now show that the residential density distribution is invariant in time. Tha is, $dn/dt = 0$ for the given technologies, preference and transportation system By equations (7.1.3), we have

$$R(\omega, t) = \frac{(r(K(t)) + \delta_k) \beta_h k_h(\omega, t) n(\omega, t)}{\alpha_h}, \quad 0 \le \omega \le L. \qquad \text{(A.7.2.7}$$

Integrate the above equation from 0 to L

$$\bar{r} = \frac{(r+\delta_k)\beta_h K_h}{\alpha_h N}. \tag{A.7.2.8}$$

As r and K_h are functions of K and m, the per-capital income from land is a function of $K(t)$ and $m(t)$ at any point of time.

From the definition of $\bar{y}(\omega)$, $\bar{m}(\omega) = \varepsilon\bar{y}(\omega)/(r+\pi)$ in (7.1.10), and $a = k + \bar{m}$, we have

$$k(\omega) = f_1(K,m)a(\omega) - f_2(K,m), \tag{A.7.2.9}$$

in which

$$f_1(K,m) \equiv \frac{r+\Omega_\pi(K,m)}{\varepsilon}, \quad f_2(K,m) \equiv \frac{w+\mu m+\bar{r}}{(r+\pi)/\varepsilon+1+r},$$

where we use

$$\pi = \Omega_\pi(K,m) \equiv \mu - \frac{\Omega_m(K,m)}{m}.$$

Substituting $s = \lambda\bar{y}$ and the definition of $\bar{y}(\omega)$ into equation (7.1.11) yields

$$\dot{a}(\omega) = (1+r)\lambda k(\omega) + \lambda w + \lambda\mu m + \lambda\bar{r} - a(\omega), \quad 0 \le \omega \le L.$$

Insert equation (A.7.2.9) into the above equation

$$\dot{a}(\omega) = (\lambda f_1 + \lambda r f_1 - 1)a(\omega) - (1+r)\lambda f_2 + \lambda w + \lambda\mu m + \lambda\bar{r}, \quad 0 \le \omega \le L. \tag{A.7.2.10}$$

This is a linear differential equation in $a(\omega,t)$. For convenience of expression, we rewrite equation (A.7.1.9) as

$$\dot{a}(\omega,t) + h_1(t)a(\omega,t) = h_2(t), \quad 0 \le \omega \le L, \tag{A.7.2.11}$$

where

$$h_1(t) \equiv 1 - \lambda f_1(K(t),m(t)) - \lambda r(K(t),m(t))f_1(K(t),m(t)),$$
$$h_1(t) \equiv -(1+r(K(t),m(t)))\lambda f_2(K(t),m(t)) + \lambda w(K(t),m(t)) + \lambda\mu m(t) + \lambda\bar{r}(K(t),m(t)).$$

Applying the general solution for this type of linear differential equations, we solve equation (A.7.2.11) as

$$a(\omega,t) = e^{-\int h_1(\tau)d\tau}\left(h_0 + \int h_2(\tau)e^{\int h_1(\tau)d\tau}d\tau\right), \tag{A.7.2.12}$$

where h_0 is a constant to be determined by initial conditions. Equation (A.7.2.12) tells that the path of households' wealth over time and space is given as a function of

the total physical wealth and real money balances of the economy. From equation (A.7.2.12), as h_1 and h_2 are independent of location, we see that if the initial distribution of the wealth is not dependent on location,[28] h_0 is also independent of the location. From equation (A.7.2.12), we conclude that $a(\omega, t)$ is independent of location, that is

$$a(\omega_1, t) = a(\omega_2, t), \quad 0 \leq \omega_1, \omega_2 \leq L.$$

By this property and equation (A.7.2.9), we conclude

$$k(\omega_1) = k(\omega_2), \quad 0 \leq \omega_1, \omega_2 \leq L.$$

From this equation and the definition of $\bar{y}(\omega)$, we have

$$\bar{y}(\omega_1) = \bar{y}(\omega_2), \quad 0 \leq \omega_1, \omega_2 \leq L.$$

From equations (7.1.10), we see that $c(\omega), \bar{m}(\omega)$ and $s(\omega)$ are invariant in space. Substituting equations (7.1.10) into $U(\omega, t)$ in (7.1.9) and then using $U(0) = U(\omega)$, we have

$$\frac{R_h(\omega)}{R_h(0)} = \left(\frac{n(\omega)}{n(0)}\right)^{\vartheta/\eta} \left(\frac{T_h(\omega)}{T_h(0)}\right)^{\sigma/\eta}, \tag{A.7.2.13}$$

where we use $\bar{y}(0) = \bar{y}(\omega)$. From $R_h(\omega)c_h(\omega) = \eta\bar{y}(\omega)$ and $(r + \delta_k)k_h = \alpha_h R_h c_h$, we have

$$k_h = \frac{\alpha_h \eta \bar{y}}{r + \delta_k}. \tag{A.7.2.14}$$

Substituting $c_h = k_h^{\alpha_h} L_h^{\beta_h}$ into $(r + \delta_k)k_h = \alpha_h c_h R_h$, we have

$$R_h(\omega) = \eta^{\beta_h} \left(\frac{r + \delta_k}{\alpha_h}\right)^{\alpha_h} n^{\beta_h}(\omega)\bar{y}^{\beta_h}. \tag{A.7.2.15}$$

Substitute equation (A.7.2.15) into equation (A.7.2.13)

$$n(\omega) = n(0)\left(\frac{T_h(\omega)}{T_h(0)}\right)^{\beta_0}, \tag{A.7.2.16}$$

where

$$\beta_0 \equiv \frac{\sigma}{\eta\beta_n - \vartheta}.$$

Integrating equation (A.7.2.16), we obtain

$$n(0, t) = \frac{NT_h^{\beta_0}(0)}{\int_0^L T_h^{\beta_0}(\omega)\, d\omega}. \tag{A.7.2.17}$$

As $T_h(\omega)$ is explicitly defined as a function of location and independent of time, we see that $n(0,t)$ is independent of time. By equation (A.7.2.16), we conclude that $n(\omega,t)$ is independent of time but dependent on location.

We now determine the differential equation for $K(t)$. From equation (A.7.2.10), $\overline{m}=m$, and $a=k+\overline{m}$, we have

$$\dot{k}=(\lambda f_1+\lambda r f_1-1)k+f_0(K,m)-(1+r)\lambda f_2+\lambda w+\lambda\bar{r}-\Omega_m, \tag{A.7.2.18}$$

where we use $\dot{m}=\Omega_m(K,m)$ and

$$f_0(K,m)\equiv(\lambda f_1+\lambda r f_1-1+\lambda\mu)m.$$

It is important to note that no variable in (A.7.2.18) is dependent on location. Multiplying (A.7.2.18) by N, we have

$$\dot{K}=\Omega_K(K,m)\equiv(\lambda f_1+\lambda r f_1-1)K+f_0N-(1+r)\lambda N f_2 + \lambda N w+\lambda N\bar{r}-N\Omega_m. \tag{A.7.2.19}$$

The two differential equations, (A.7.2.19) and (A.7.2.6), contain two location-independent variables, $K(t)$ and $m(t)$.

8 Money in multi-regional and growth economies

One of the most salient features of the geography of economic activity is concentration. Over the course of economic development, the patterns of regional agglomeration become more pronounced: different regions grow at differential rates, and a few metropolitan areas continue to attract people while others remain stagnant or even witness decline in their population figures. It is observed by Bairoch (1993) that great disparities between rich and poor regions are recent. It is important to know why and how they change over time. The emergence of large cities is a fact that Kuznets (1966) associates with industrialization. The link between industrialization and agglomeration has long been recognized by economists. The development theories by Myrdal (1957) and Hirschman (1958) identify a fundamental interdependence between industrial growth and the geographical concentration of industry: industrialization collects human and capital resources in a given location and the resulting agglomeration sustains the conditions for further growth. Recent literature on economic geography formalizes such cumulative processes in models based on increasing returns to scale.[1] The strong positive correlation between industrialization and geographic clustering of economic activities has been observed by empirical studies as well.[2] Nevertheless, the new economic geography has not succeeded in including capital accumulation and infrastructures formation as endogenous processes of industrialization and agglomeration. This chapter concentrates on the study of interregional development with capital accumulation, taking account of factors such as environment and regional economic structure.

It is significant to explain economic geography with capital accumulation. Capital accumulation is one of the key dynamic forces of economic growth and development. Modern cities and metropolitan areas are characterized not only of people and knowledge, but also of concentration of buildings, infrastructures, and machines. Yet only a few attempts have been made to introduce capital accumulation into formal dynamic economic geography with rational assumptions of profit and utility maximization. As argued by Zhang (2005a), a main obstacle

to properly modeling geographical dynamics with capital accumulation is that the traditional approaches to consumer behavior over time makes it analytically intractable to model spatial growth with rational consumer behavior. This chapter uses an alternative approach to consumer behavior in modeling a national economic growth with two regions. We demonstrate that the dynamic regional equilibrium model based on the new utility function approach is analytically tractable.

This chapter analyzes a perfectly competitive economy with regional differentials in living conditions and productivity. The productivity advantages of one region may be offset to some extent by the higher wages that must be paid in a system where people are free to choose where they work and live. Higher wages are often associated with some kinds of disamenities (such as noise, pollutants, and densely populated neighborhood) and high living costs. Labor force, capital, and knowledge are main inputs of economic production. Labor and capital are easily mobile between regions in industrialized economies. As capital mobility becomes high and costs associated with capital movement among regions become low, it is reasonable to assume that capital movement tends to equalize marginal productivities of capital among regions within a national economy. But there are different principles for analyzing temporary equilibrium conditions for labor movement in a dynamic regional framework. For example, in the theoretical literature on regional economics two principles have been proposed to analyze labor movement. One is that labor movement, if costs associated with professional or locational changes are neglected, equalizes wage rates between regions. This assumption is limited in a freely competitive national labor market if various regions provide different levels of amenity (such as regional cultures, climates, and pollution) and have different technologies. The other one is that free movement of people equalizes utility levels, which they may obtain in different regions. It should be noted that this approach is invalid in the growth theory based on the Solow model as the theory does not use utility function in analyzing consumer behavior. Although the assumption of equalizing utility level is reasonable for analyzing issues related to movement of people among regions, the traditional approaches to consumer behavior in economic theory often make it analytically intractable to properly model migration with regional differences in life styles and wage rates. If wage rates and life style (including saving behavior) are regionally different and people may freely move within the economy, how to calculate each individual's capital and his income in a dynamic multi-regional economy is not an easy matter. In this chapter, we solve this issue by utilizing the alternative utility function. We also attempts to introduce amenity into the neoclassical growth theory in a regional context. It is well observed that a country in the beginning of its economic development will be experiencing a worsening of the environment, while a country in which growth has taken place over a longer period of time will be adjusting its patterns of growth in such a way that the environment in fact improves.

This chapter is a generalization of the two-region growth model proposed by Zhang (1998b).[3] This chapter generalizes the previous model in different aspects.

The previous study only analyzes some properties of the model without simulation; this study provides simulation results. This study treats regional amenity as endogenous variables; while the previous study considers amenity as exogenous variable. Moreover, as far as the interregional economic issues are concerned, this chapter is similar to the GTAP model which is a multiregional, multisector, computable general equilibrium model, with perfect competition and constant returns to scale.[4] Attempts have been made to extend the standard static GTAP model to dynamic model.[5] There are many other models developed by regional scientists. For instance, some models based on the input-output system or/and the gravity theory are proposed to examine interregional trade patterns.[6] This chapter differs from these approaches in that we model behavior of households differently.

8.1 The two-region monetary growth model

This section builds a dynamic two-region and two-sector trade model to examine interdependence between regional trades and national growth. We analyze trade issues within the framework of a simple international macroeconomic growth model with perfect capital mobility. This model is influenced by the neoclassical trade theory with capital accumulation.[7] Most of neoclassical interregional growth models with capital accumulation assume that all the regions produce a homogenous commodity. This study extends the model by explicitly taking account of services.

Each region produces one goods and services. Most aspects of the goods sector in our model are similar to the neo-classical one-sector growth model. There is only one (durable) good in the national economy under consideration. Households own assets of the economy and distribute their incomes to consume and save. Production sectors or firms use capital and labor. Exchanges take place in perfectly competitive markets. Production sectors sell their product to households or to other sectors and households sell their labor and assets to production sectors. Factor markets work well; factors are inelastically supplied and the available factors are fully utilized at every moment. Saving is undertaken only by households, which implies that all earnings of firms are distributed in the form of payments to factors of production. We omit the possibility of hoarding of output in the form of non-productive inventories held by households. All savings volunteered by households are absorbed by firms. We require saving and investment to be equal at any point of time.

The system consists of two regions, indexed by $j = 1, 2$. Perfect competition is assumed to prevail in all the markets both within each region and between the regions, and commodities are traded without any barriers such as transport costs or tariffs. The labor markets are perfectly competitive within each region and between regions. Let prices be measured in terms of the commodity and the price of the commodity be unity. We denote wage and interest rates by $w_j(t)$ and $r_j(t)$, respectively, in the j th region. The interest rate is identical throughout the national economy, i.e. $r(t) = r_j(t)$. We assume a homogenous population. A person is free

to choose his residential location. We assume that any person chooses the same region where he works and lives. Each region has fixed land. Land quality, climates, and environment are homogenous within each region, but they may vary among the regions. We neglect transportation cost of commodities between and within regions. As will become evident later on, although it is conceptually not difficult to introduce transportation cost function and to provide balance conditions for demand and supply and for price equalization conditions with transportation cost,[8] the problem will become analytically too complicated. By the way, McCann (2005) points out some problems with the well-accepted assumptions on transportation costs in the literature of the new economic geography. The assumption of zero transportation cost of commodities implies price equality for the commodity among regions. As amenity and land are immobile, wage rates and land rent may not be equal among regions.

We introduce

N: the given population of the economy;
i, s: subscript index for the industrial and services sectors, respectively;
L_j: the given (residential) area of region j;
$K(t)$: the total capital stocks of the economy at time t;
$F_{jq}(t)$: the output levels of q's sector in region j at time t, $q = i, s$;
$K_{jq}(t)$ and $N_{jq}(t)$: the levels of capital stocks and labor force employed by sector q in region j;
$c_{jq}(t)$ and $s_j(t)$: per capita's consumption level of product q and savings made by per capita in region j;
$l_j(t)$: the lot size per capita in region j; and
$p_j(t)$ and $R_j(t)$: region j's services price and land rent, respectively.

Behavior of producers

We assume that there are only two productive factors, capital, $K_{jq}(t)$, and labor, $N_{jq}(t)$, at each point of time t. The production functions are given by

$$F_{jq}\left(K_{jq}(t),\ N_{jq}(t)\right), \quad j = 1, 2,\ q = i, s,$$

where F_{jq} is the output of sector q in region j. Assume F_{jq} to be neoclassical. We have

$$f_{jq}(t) = f_{jq}\left(k_{jq}(t)\right), \quad f_{jq}(t) \equiv \frac{F_{jq}(t)}{N_{jq}(t)}, \quad k_{jq}(t) \equiv \frac{K_{jq}(t)}{N_{jq}(t)}. \tag{8.1.1}$$

Markets are competitive; thus labor and capital earn their marginal products, and firms earn zero profits. The rate of interest, $r(t)$, and wage rates, $w_j(t)$, are determined by markets. Hence, for any individual firm $r(t)$ and $w_j(t)$ are given at

each point of time. The production sector chooses the two variables, $K_{jq}(t)$ and $N_{jq}(t)$, to maximize its profit. The marginal conditions are given by

$$r+\delta_{kj}=f_{ji}'\left(k_{ji}\right),\ w_j(t)=f_{ji}\left(k_{ji}\right)-k_{ji}f_{ji}'\left(k_{ji}\right),$$
$$r+\delta_{kj}=p_jf_{js}'\left(k_{js}\right),\ w_j(t)=p_j\left[f_{js}\left(k_{js}\right)-k_{js}f_{js}'\left(k_{js}\right)\right], \tag{8.1.2}$$

where δ_{kj} is the depreciation rate of physical capital in region j.

Behavior of consumers

Each worker may get income from land ownership, wealth ownership and wages. In order to define incomes, it is necessary to determine land ownership structure. It can be seen that land properties may be distributed in multiple ways under various institutions. To simplify the model, we accept the assumption of absent landownership, which means that the income of land rent is spent outside the economic system. A possible reasoning for this is that the land is owned by the government, people can rent the land in competitive market, and the government uses the income for military or other public purposes. In the appendix, we analyze the dynamics in a different situation when the land is equally owned by the households.

Let $P(t), \mu, M(t)$, and $m(t)$ be defined as in Section 2.2. With μ being the constant net growth rate of the per-capita money stock, $M(t)$, evolves over time according to

$$\dot{M}(t)=\mu M(t),\quad \mu>0.$$

The government expenditure in real terms per capita, $\tau(t)$, is given by

$$\tau(t)=\frac{\dot{M}(t)}{P(t)}=\frac{\mu M(t)}{P(t)}=\mu m(t).$$

The representative household receives $\mu m(t)$ units of paper money from the government though a "helicopter drop," also considered to be independent of his money holdings. Consumers make decisions on choice of lot size, consumption level of commodity as well as on how much to save. Let $m_j(t)$ and $\overline{k}_j(t)$ stand for respectively the per capita real money balances and the per capita physical wealth (excluding land) in region j. Each consumer of region j obtains the current income

$$y_j(t)=r(t)\overline{k}_j(t)+w_j(t)-\pi(t)m_j(t)+\mu m(t),\quad j=1,2. \tag{8.1.3}$$

The disposable income is given by

$$\hat{y}_j(t)=y_j(t)+a_j(t), \tag{8.1.4}$$

where $a_j(t)\equiv\overline{k}_j(t)+m_j(t)$.

At each point of time, a consumer distributes the total available budget among real money balances, $m_j(t)$, consuming lot size, $l_j(t)$, saving, $s_j(t)$, consumption of goods, $c_j(t)$. The budget constraint is given by

$$(1+r)m_j + R_j l_j + c_{ji} + p_j c_{js} + s_j = \hat{y}_j.$$

From this equation and equations (8.1.3) and (8.1.4), we have

$$(\pi + r)m_j + R_j l_j + c_{ji} + p_j c_{js} + s_j = \bar{y}_j \equiv (1+r)\bar{k}_j + w_j + \mu m. \tag{8.1.5}$$

The utility level of the typical consumer in region j is represented by

$$U_j(t) = \theta_j(t) m_j^{\varepsilon_0}(t) l_j^{\eta_0}(t) c_{ji}^{\xi_0}(t) c_{js}^{\gamma_0}(t) s_j^{\lambda_0}(t), \quad \varepsilon_0, \eta_0, \gamma_0, \xi_0, \lambda_0 > 0, \tag{8.1.6}$$

in which $\varepsilon_0, \eta_0, \xi_0, \gamma_0$, and λ_0 are a typical person's elasticity of utility with regard to real money balances, lot size, commodity and savings. We call η_0, ξ_0, γ_0, and λ_0 propensities to hold money, to consume lot size, goods, and services, and to hold wealth (save), respectively. We call $\theta_j(t)$ region j's amenity level. This variable is defined similarly as in Chapter 7. We assume that amenity is affected by production and consumption activities. We specify θ_j as follows

$$\theta_j(t) = \bar{\theta}_j F_{ji}^a(t) F_{js}^b(t) N_j^d(t), \quad j = 1, 2,$$

where $\bar{\theta}_j (> 0)$, a, d, and b are parameters and $N_j(t)$ is region j's population. We do not specify signs of a, d, and b as economic activities and population may have either positive or negative effects on regional attractiveness. Regional amenities are given to individuals. As $F_{jq} = f_{jq} N_{jq}$, we can rewrite the above equations as follows

$$\theta_j(t) = \bar{\theta}_j f_{ji}^a(t) f_{js}^b(t) N_{ji}^a(t) N_{js}^b(t) N_j^d(t), \quad j = 1, 2. \tag{8.1.7}$$

Maximizing $U_j(t)$ subject to budget constraints (8.1.5) yields

$$(\pi + r)m_j = \varepsilon \bar{y}_j, \quad l_j R_j = \eta \bar{y}_j, \quad c_{ji} = \xi \bar{y}_j, \quad p_j c_{js} = \gamma \bar{y}_j, \quad s_j = \lambda \bar{y}_j, \tag{8.1.8}$$

where

$$\varepsilon \equiv \rho \varepsilon_0, \quad \eta \equiv \rho \eta_0, \quad \xi \equiv \rho \xi_0, \quad \gamma \equiv \rho \gamma_0, \quad \lambda \equiv \rho \lambda_0,$$

$$\rho \equiv \frac{1}{\varepsilon_0 + \eta_0 + \xi_0 + \gamma_0 + \lambda_0}.$$

According to the definitions of $s_j(t)$, the wealth accumulation of the representative person in region j is given by

$$\dot{a}_j(t) = s_j(t) - a_j(t). \tag{8.1.9}$$

As households are assumed freely mobile among the regions, the utility level of people should be equal, irrespective of which region they live in, i.e.

$$U_1(t) = U_2(t).$$

The national real balance is equal to the sum of the real money balances of all the households. That is,

$$\sum_{j=1}^{2} m_j(t)N_j(t) = Nm(t). \tag{8.1.10}$$

The total capital stocks employed by the production sectors are equal to the total wealth owned by all the regions. That is,

$$K(t) = \sum_{j=1}^{2} K_j(t) = \sum_{j=1}^{2} \bar{k}_j(t)N_j(t), \tag{8.1.11}$$

in which

$$K_j(t) \equiv K_{ji}(t) + K_{js}(t).$$

A region's supply of services is only consumed by the region

$$c_{js}(t)N_j(t) = F_{js}(t), \quad j = 1, 2. \tag{8.1.12}$$

We introduce the national consumption of the industrial goods, $C_i(t)$, the total savings, $S(t)$, and the total output levels of the industrial sector as

$$C_i(t) = \sum_{j=1}^{2} c_{ji}(t)N_j(t),\ S(t) = \sum_{j=1}^{2} s_j(t)N_j(t),\ F_i(t) = \sum_{j=1}^{2} F_{ji}(t).$$

We assume that labor force and land are fully employed. The assumption of full employment of production factors is represented as

$$N_{ji}(t) + N_{js}(t) = N_j(t),\ \sum_{j=1}^{2} N_j(t) = N,\ l_j(t)N_j(t) = L_j, \quad j = 1, 2. \tag{8.1.13}$$

We have thus built the model, which explains endogenous capital accumulation and regional money, capital and labor distribution in the national economy in which all the markets are perfectly competitive and product, capital and labor are freely mobile.

8.2 The dynamics of the non-monetary economy

The general dynamic system is complicated. A thorough examination of the general dynamic system is too difficult. For illustration, this section simulates the non-monetary model, that is, the case of $\varepsilon_0 = 0$. Before simulating the model, we determine the procedure of how to calculate all the variables at each point of time with given initial conditions.

8.2.1 The motion of the non-monetary economy

For simplicity, this section specifies the production functions in the Cobb-Douglass form. In Appendix 8.1, we show that the dynamics of the national economy can be expressed as 2-dimensional differential equations for general forms of the production functions. We specify the production functions as follows

$$F_{jq}(t) = A_{jq}K_{jq}^{\alpha_{jq}}(t)N_{jq}^{\beta_{jq}}(t),\ \alpha_{jq} + \beta_{jq} = 1,\ \alpha_{qj},\ \beta_{jq} > 0,\ j = 1,2,\ q = i,s, \tag{8.2.1}$$

where A_{jq} is sector q's productivity in region j. From $f_{jq} = A_{jq}k_{jq}^{\alpha_{jq}}$, and equations (8.1.2), we have

$$\begin{aligned} &r + \delta_{kj} = \alpha_{ji}A_{ji}k_{ji}^{-\beta_{ji}}, \quad w_j = \beta_{ji}A_{ji}k_{ji}^{\alpha_{ji}}, \\ &r + \delta_{kj} = \alpha_{js}A_{js}p_jk_{js}^{-\beta_{js}}, \quad w_j = \beta_{si}A_{js}p_jk_{js}^{\alpha_{js}}. \end{aligned} \tag{8.2.2}$$

As shown in Appendix A.8.1, we can express k_{2i}, k_{js}, w_j, and p_j as functions of k_{1i} as follows

$$\begin{aligned} &k_{2i} = \left(\frac{\alpha_{2i}A_{2i}}{\alpha_{1i}A_{1i}k_{1i}^{-\beta_{1i}} + \delta}\right)^{1/\beta_{2i}}, \quad w_j = \phi_j(k_{1i}) \equiv \beta_{ji}A_{ji}\phi_{ji}^{\alpha_{ji}}, \\ &r = \phi_r(k_{1i}) \equiv \alpha_{1i}A_{1i}k_{1i}^{-\beta_{1i}} - \delta_{k1}, \quad k_{js} = \phi_{js}(k_{1i}) \equiv \left(\frac{\alpha_{js}}{\beta_{si}}\right)\frac{\phi_j(k_{1i})}{\delta_{kj} + \phi_r(k_{1i})}, \\ &j = 1,2, \\ &p_j = \varphi_j(k_{1i}) \equiv \frac{\phi_j}{\beta_{js}A_{js}k_{js}^{\alpha_{js}}}, \end{aligned} \tag{8.2.3}$$

in which $\phi_{1i}(k_{1i}) \equiv k_{1i}$ and $\delta \equiv \delta_{k2} - \delta_{k1}$. Also by $f_{jq} = A_{jq}k_{jq}^{\alpha_{jq}}$, we see that f_{jq} are functions of k_{1i}. In the non-monetary economy, $\bar{y}_j$ is the same as $\hat{y}_j$. By equations (8.1.3), (8.1.4), and (8.2.3), we have

$$\hat{y}_j = \bar{\phi}_j\left(k_{1i}, \bar{k}_j\right) \equiv (1 + \phi_r)\bar{k}_j + \phi_j. \tag{8.2.4}$$

From equations (8.1.12), we have

$$c_{js}N_j = N_{js}f_{js}, \quad j = 1,2.$$

In this section, we always require $\eta_0 + \gamma_0 + \xi_0 + \lambda_0 = 1$. We have $\eta = \eta_0, \gamma = \gamma_0, \xi = \xi_0$, and $\lambda = \lambda_0$. Substitute $p_j c_{js} = \gamma \hat{y}_j$ into the above equations

$$N_{js} = \frac{\gamma \hat{y}_j N_j}{p_j f_{js}}, \quad j = 1, 2. \tag{8.2.5}$$

From these equations and $N_{ji} + N_{js} = N_j$, we have

$$N_{ji} = \left(f_{js} - \frac{\gamma \hat{y}_j}{p_j} \right) \frac{N_j}{f_{js}}, \quad j = 1, 2. \tag{8.2.6}$$

Equations (8.2.5) and (8.2.6) determine the labor distribution between the two sectors as functions of the regional labor force, the disposable income, and k_{1i} at each point of time.

In this section, we assume $a = 0$. We omit possible effects of the industrial sector on the region's amenity. This is mainly for simplifying our analysis. We discuss the case when a is not equal to zero in Appendix. From Appendix, we also see why the assumption of $a = 0$ simplifies the analysis. Substitute equations (8.2.5) into equations (8.1.7)

$$\theta_j = \overline{\theta}_j \gamma^b p_j^{-b} \hat{y}_j^b N_j^{d+b}.$$

Substitute the above equations, $l_j = L_j / N_j$ and equations (8.1.8) into utility functions

$$U_j = \overline{\theta}_j \xi^\xi \gamma^{b+\gamma} \lambda^\lambda L_j^\eta p_j^{-b-\gamma} \hat{y}_j^{b+\xi+\gamma+\lambda} N_j^{d+b-\eta}.$$

By these equations and $U_1 = U_2$, we have

$$n = p(k_{1i}) \left(\frac{\hat{y}_1}{\hat{y}_2} \right)^{\overline{\eta}}, \tag{8.2.7}$$

in which

$$n \equiv \frac{N_2}{N_1}, \; p(k_{1i}) \equiv \left(\frac{\overline{\theta}_1 L_1^\eta}{\overline{\theta}_2 L_2^\eta} \right)^{1/(d+b-\eta)} \left(\frac{p_2}{p_1} \right)^{(b+\gamma)/(d+b-\eta)}, \; \overline{\eta} \equiv \frac{b+\xi+\gamma+\lambda}{d+b-\eta}.$$

From equation (8.1.11), we have

$$\sum_{j=1}^{2} \left(k_{ji} N_{ji} + k_{js} N_{js} \right) = \sum_{j=1}^{2} \overline{k}_j N_j. \tag{8.2.8}$$

Insert equations (8.2.5) and (8.2.6) into equation (8.2.8)

$$\sum_{j=1}^{2} \left(k_{ji} - \left(k_{ji} - k_{js} \right) \frac{\gamma \hat{y}_j}{p_j f_{js}} \right) N_j = \sum_{j=1}^{2} \overline{k}_j N_j. \tag{8.2.9}$$

We solve equation (8.2.9) in $\overline{k}_1$

$$\overline{k}_1 = -n\overline{k}_2 + \sum_{j=1}^{2}\left(k_{ji} - (k_{ji} - k_{js})\frac{\gamma\hat{y}_j}{p_j f_{js}}\right)\frac{N_j}{N_1}. \tag{8.2.10}$$

By equations (8.2.4), we obtain

$$\overline{k}_1 = \frac{\hat{y}_1 - \phi_1}{1+\phi_r}. \tag{8.2.11}$$

Equal the right-hand sides of equations (8.2.10) and (8.2.11)

$$-n\overline{k}_2 + \sum_{j=1}^{2}\left(k_{ji} - (k_{ji} - k_{js})\frac{\gamma\hat{y}_j}{p_j f_{js}}\right)\frac{N_j}{N_1} = \frac{\hat{y}_1 - \phi_1}{1+\phi_r}.$$

Insert $n = p(k_{1i})(\hat{y}_1/\hat{y}_2)^{\overline{\eta}}$ into the above equation

$$\hat{y}_1 - \overline{f}\left(k_{1i}, \overline{k}_2\right)\hat{y}_1^{\overline{\eta}} - \hat{f}\left(k_{1i}, \overline{k}_2\right) = 0, \tag{8.2.12}$$

in which

$$\hat{f}\left(k_{1i}, \overline{k}_2\right) \equiv \left[\frac{\phi_1}{1+\phi_r} + k_{1i}\right]\frac{1}{\overline{f}_0\left(k_{1i}, \overline{k}_2\right)},$$

$$\overline{f}\left(k_{1i}, \overline{k}_2\right) \equiv \left(-\overline{k}_2 + k_{2i} - (k_{2i} - k_{2s})\frac{\gamma\hat{y}_2}{p_2 f_{2s}}\right)\frac{p}{\overline{f}_0\left(k_{1i}, \overline{k}_2\right)\hat{y}_2^{\overline{\eta}}},$$

$$\overline{f}_0(k_{1i}) \equiv \frac{1}{1+\phi_r} + (k_{1i} - k_{1s})\frac{\gamma}{p_1 f_{1s}}.$$

By $\phi_r + \delta_{k1} = \alpha_{1s} p_1 f_{1s}/k_{1s}$ and the definition of $\overline{f}_0(k_{1i})$, we have

$$\overline{f}_0\left(k_{1i}, \overline{k}_2\right) = \frac{(1-\alpha_{1s}\gamma)\phi_r + \delta_{k1} - \alpha_{1s}\gamma}{(1+\phi_r)(\phi_r + \delta_{k1})} + \frac{\alpha_{1s}\gamma k_{1i}}{(\phi_r + \delta_{k1})k_{1s}}.$$

As $1 - \alpha_{1s}\gamma > 0$, we see that if $\delta_{k1} - \alpha_{1s}\gamma \geq 0$, then $\overline{f}_0(k_{1i}) > 0$. It can be seen that it is not restricted to require $\delta_{k1} - \alpha_{1s}\gamma > 0$. For instance, if we specify $\delta_{k1} = 0.05$ and $\alpha_{1s} = 1/3$, then the inequality is satisfied if $0.15 \geq \gamma$. In this study, we assume $\overline{f}_0(k_{1i}) > 0$. As $\overline{f}_0(k_{1i}) > 0$, we have $\hat{f} > 0$. These properties are important, as shown later on, for guaranteeing a unique solution of the equation when we simulate the model.

It should be noted that the functions $\hat{f}$ and $\overline{f}$ are only dependent on 2 variables, $k_{1i}(t)$ and $\overline{k}_2(t)$. This is important as it will allow us to express the dynamics in

a 2-dimensional differential equations system. It is impossible to explicitly solve equation (8.2.12). We assume that equation (8.2.12) has a (at least one) positive solution, denoted as

$$\hat{y}_1 = \Lambda_0\left(k_{1i}, \overline{k}_2\right). \tag{8.2.13}$$

This equation is crucial for finding out explicit expressions of the dynamics. In general, equation (8.2.12) may have multiple solutions for any given positive $k_{1i}(t)$ and $\overline{k}_2(t)$, depending on the parameter value of $\overline{\eta}$. By equation (8.2.13), we express $\hat{y}_1$ as a function of $k_{1i}(t)$ and $\overline{k}_2(t)$. Substitute (8.2.13) into equation (8.2.11)

$$\dot{\overline{k}}_1 = \Lambda\left(k_{1i}, \overline{k}_2\right) \equiv \frac{\Lambda_0\left(k_{1i}, \overline{k}_2\right) - \phi_1(k_{1i})}{1 + \phi_r(k_{1i})}. \tag{8.2.14}$$

Substitute $s_j = \lambda \hat{y}_j$ from equations (8.1.8) into equations (8.1.9)

$$\dot{\overline{k}}_j = \lambda \hat{y}_j - \overline{k}_j.$$

Substitute equations (8.2.4), (8.2.13), and (8.2.14) into the above equations

$$\dot{\overline{k}}_1 = \Lambda_1\left(k_{1i}, \overline{k}_2\right) \equiv \lambda\Lambda_0\left(k_{1i}, \overline{k}_2\right) - \Lambda\left(k_{1i}, \overline{k}_2\right), \tag{8.2.15}$$

$$\dot{\overline{k}}_2 = \Lambda_2\left(k_{1i}, \overline{k}_2\right) \equiv (\lambda\phi_r(k_{1i}) + \lambda - 1)\overline{k}_2 + \lambda\phi_j(k_{1i}). \tag{8.2.16}$$

Taking derivatives of equation (8.2.14) with respect to t yields

$$\dot{\overline{k}}_1 = \frac{\partial\Lambda}{\partial k_{1i}}\dot{k}_{1i} + \Lambda_2\frac{\partial\Lambda}{\partial\overline{k}_2}, \tag{8.2.17}$$

where we use equation (8.2.16). We do not explicitly calculate partial derivatives as it is straightforward but the expressions are tedious. Equating both right-hand sides of equations (8.2.15) and (8.2.17), we solve

$$\dot{k}_{1i} = \frac{\lambda\Lambda_0\left(k_{1i}, \overline{k}_2\right) - \Lambda\left(k_{1i}, \overline{k}_2\right) - \Lambda_2\partial\Lambda/\partial\overline{k}_2}{\partial\Lambda/\partial k_{1i}}. \tag{8.2.18}$$

We see that $\dot{k}_{1i}$ is a function of $k_{1i}(t)$ and $\overline{k}_2(t)$. From equation (8.2.16), $\dot{\overline{k}}_2$ is a function of $k_1(t)$ and $\overline{k}_2(t)$. In summary, we obtain Lemma 8.2.1.

Lemma 8.2.1

The dynamics of the national economy is given by the following 2-dimensional differential equations with $k_{1i}(t)$ and $\bar{k}_2(t)$ as the variables

$$\dot{k}_{1i} = \frac{\lambda\Lambda_0\left(k_{1i}, \bar{k}_2\right) - \Lambda\left(k_{1i}, \bar{k}_2\right) - \Lambda_2 \partial\Lambda/\partial\bar{k}_2}{\partial\Lambda/\partial k_{1i}},$$

$$\dot{\bar{k}}_2 = \Lambda_2\left(k_{1i}, \bar{k}_2\right). \tag{8.2.19}$$

For any given positive values of $k_{1i}(t)$ and $\bar{k}_2(t)$ at any point of time, the other variables are uniquely determined by the following procedure: $\hat{y}_1$ by (8.2.13) $\rightarrow$ $\bar{k}_1$ by (8.2.14) $\rightarrow r, p_j, w_j, k_{2i}, k_{js}, j = 1, 2$ by (8.2.3) $\rightarrow f_{jq} = A_{jq}k_{jq}^{\alpha_{jq}}, q = i, s \rightarrow \hat{y}_2$ by (8.2.2) $\rightarrow n$ by (8.2.7) $\rightarrow N_1 = N/1+n \rightarrow N_2 = nN_1 \rightarrow l_j = L_j/N_j \rightarrow R_j(t) = \eta\hat{y}_j/l_j \rightarrow c_{ji}, c_{js}$, and s_j by (8.1.8) $\rightarrow N_{js}$ by (8.2.5) $\rightarrow N_{ji}$ by (8.2.6) $\rightarrow F_{jq} = N_{jq}f_{jq}$.

The 2-dimensional differential equations (8.2.19) contain 2 variables, $k_{1i}(t)$ and $\bar{k}_2(t)$. Rather than $\bar{k}_1(t)$, we use $k_{1i}(t)$ as a variable in the ultimate system for dynamic analysis. The replacement makes our dynamic analysis tractable. We see that $k_{1i}(t)$ and its derivative play a reference role in the technique to find and express a solution to the whole dynamic system. With proper initial conditions, the system determines the values of these variables at any point of time. Following Lemma 8.2.1, we can determine all the other variables in the national economic system.

8.2.2 The steady state of the non-monetary economy

An equilibrium point is given by setting $\dot{k}_{1i} = 0$ and $\dot{k}_2 = 0$ in equations (8.2.19). That is

$$\lambda\Lambda_0\left(k_{1i}, \bar{k}_2\right) - \Lambda\left(k_{1i}, \bar{k}_2\right) = 0, \tag{8.2.20}$$

$$(\lambda\phi_r(k_{1i}) + \lambda - 1)\bar{k}_2 + \lambda\phi_2(k_{1i}) = 0. \tag{8.2.21}$$

From equation (8.2.21), we solve

$$\bar{k}_2 = \frac{\lambda\phi_2(k_{1i})}{1 - \lambda - \lambda\phi_r(k_{1i})}. \tag{8.2.22}$$

Substitute equations (8.2.14) and (8.2.22) into equation (8.2.20)

$$\Omega(k_{1i}) \equiv (\lambda + \lambda\phi_r(k_{1i}) - 1)\Lambda_0(k_{1i}) + \phi_1(k_{1i}) = 0. \tag{8.2.23}$$

Equation (8.2.23) determines the equilibrium value of k_{1i} and equation (8.2.22) determines the equilibrium value of $\bar{k}_2$.

To simulate the model, we specify parameter values. To solve equation (8.2.12) in $\hat{y}_1$, we specify

$$\eta = 0.07, b = 0, \; d = -0.395. \tag{8.2.24}$$

The condition, $\eta = 0.07$, implies that 7 percent of the available income is spent on housing. We neglect impact of the service production on the region's amenity level, that is, $b = 0$. The condition, $d = -0.395$, implies that the population is negatively related to the region's amenity level. The higher the population, the lower the region's amenity level. As shown below, the specification $d = -0.395$ makes the problem solvable. Under (8.2.24), we have

$$\overline{\eta} = \frac{b + 1 - \eta}{d + b - \eta} = -2.$$

Hence, equation (8.2.12) becomes

$$\hat{y}_1^3 - \hat{f}\left(k_{1i}, \overline{k}_2\right)\hat{y}_1^2 - \overline{f}\left(k_{1i}, \overline{k}_2\right) = 0. \tag{8.2.25}$$

As $\hat{f}$ is positive, equation (8.2.25) has a unique solution as follows

$$\hat{y}_1\left(k_{1i}, \overline{k}_2\right) = \Lambda_0\left(k_{1i}, \overline{k}_2\right) = \frac{\hat{f}}{3} + \frac{\sqrt[3]{2}\hat{f}^2}{3\widetilde{F}} + \frac{\widetilde{F}}{3\sqrt[3]{2}},$$

in which

$$\widetilde{F}\left(k_{1i}, \overline{k}_2\right) \equiv \left[2\hat{f}^3 + 27\overline{f} + \sqrt{27\overline{f}\left(4\hat{f}^3 + 27\overline{f}\right)}\right]^{1/3}.$$

We specify the other parameters as follows

$$\begin{pmatrix} A_{1i} \\ A_{1s} \end{pmatrix} = \begin{pmatrix} 1.8 \\ 1.5 \end{pmatrix}, \begin{pmatrix} A_{2i} \\ A_{2s} \end{pmatrix} = \begin{pmatrix} 1.6 \\ 1.3 \end{pmatrix}, \begin{pmatrix} \alpha_{1i} \\ \alpha_{1s} \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.32 \end{pmatrix}, \begin{pmatrix} \alpha_{2i} \\ \alpha_{2s} \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.32 \end{pmatrix},$$

$$\begin{pmatrix} \delta_{k1} \\ \delta_{k2} \end{pmatrix} = \begin{pmatrix} 0.05 \\ 0.05 \end{pmatrix}, \begin{pmatrix} \overline{\theta}_1 \\ \overline{\theta}_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} \lambda \\ \xi \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.1 \end{pmatrix}, \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad N = 10. \tag{8.2.26}$$

As $\eta + \xi + \gamma + \lambda = 1$, we have $\gamma = 0.08$. For the both sectors, region 1's level of productivity is higher than region 2's. It should be remarked that some empirical studies on the US economy demonstrate that the value of the parameter, α, in the Cobb-Douglas production is approximately equal to 0.3. We specify values of α_{jk} close to 0.3. With regard to the technological parameters, for illustration what are important in our interregional study are their relative values. The presumed

productivity differences among the regions are not very large. It can be seen that the specified values of the land sizes, the preference parameters and the population will not affect our main concerns about interactions between the two regions. In the remainder of this study, we call region 1 as (technologically) advanced region and region 2 less advanced region.

First, we solve equation (8.2.23). We now find out numerical solution(s). We plot the function $\Omega(k_{1i})$ as in Figure 8.2.1. The equation $\Omega(k_1) = 0$ has a unique positive solution

$$k_{1i} = 8.87487.$$

From $k_{1i} = 8.875$ and equations (8.2.23), we get the equilibrium values of $\overline{k}_2$. Following Lemma 8.2.1, we solve the equilibrium values of all the other variables. We list the simulation results as follows

$$r = 0.067,\ F = 33.38,\ K = 87.09,$$

$$\begin{pmatrix} f_{1i} \\ f_{2i} \end{pmatrix} = \begin{pmatrix} 3.47 \\ 2.93 \end{pmatrix}, \begin{pmatrix} F_{1i} \\ F_{2i} \end{pmatrix} = \begin{pmatrix} 18.00 \\ 6.09 \end{pmatrix}, \begin{pmatrix} f_{1s} \\ f_{2s} \end{pmatrix} = \begin{pmatrix} 3.11 \\ 2.55 \end{pmatrix}, \begin{pmatrix} F_{1s} \\ F_{2s} \end{pmatrix} = \begin{pmatrix} 6.05 \\ 1.99 \end{pmatrix},$$

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 24.95 \\ 8.43 \end{pmatrix}, \begin{pmatrix} k_{1i} \\ k_{2i} \end{pmatrix} = \begin{pmatrix} 8.88 \\ 7.50 \end{pmatrix}, \begin{pmatrix} k_{1s} \\ k_{2s} \end{pmatrix} = \begin{pmatrix} 9.75 \\ 8.24 \end{pmatrix}, \begin{pmatrix} N_{1i} \\ N_{2i} \end{pmatrix} = \begin{pmatrix} 5.20 \\ 2.08 \end{pmatrix},$$

$$\begin{pmatrix} N_{1s} \\ N_{2s} \end{pmatrix} = \begin{pmatrix} 1.95 \\ 0.78 \end{pmatrix}, \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} 7.14 \\ 2.86 \end{pmatrix}, \begin{pmatrix} K_{1i} \\ K_{2i} \end{pmatrix} = \begin{pmatrix} 46.13 \\ 15.59 \end{pmatrix}, \begin{pmatrix} K_{1s} \\ K_{2s} \end{pmatrix} = \begin{pmatrix} 18.97 \\ 6.41 \end{pmatrix},$$

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 65.10 \\ 21.99 \end{pmatrix}, \begin{pmatrix} \overline{k}_1 \\ \overline{k}_2 \end{pmatrix} = \begin{pmatrix} 9.11 \\ 7.70 \end{pmatrix}, \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} 0.42 \\ 1.40 \end{pmatrix}, \begin{pmatrix} c_{1i} \\ c_{2i} \end{pmatrix} = \begin{pmatrix} 1.21 \\ 1.03 \end{pmatrix},$$

$$\begin{pmatrix} c_{1s} \\ c_{2s} \end{pmatrix} = \begin{pmatrix} 0.85 \\ 0.70 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2.43 \\ 2.05 \end{pmatrix}, \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1.15 \\ 1.18 \end{pmatrix}, \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} 2.03 \\ 0.51 \end{pmatrix}, \tag{8.2.27}$$

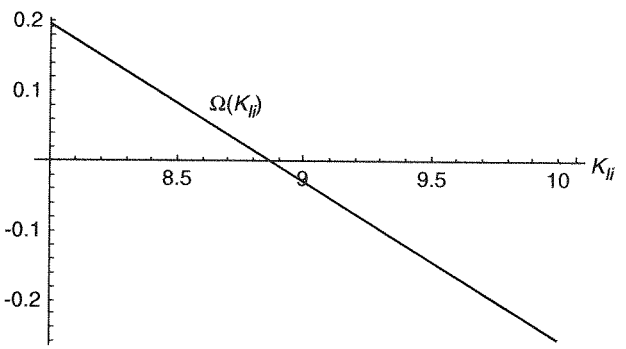


Figure 8.2.1 The existence of a unique positive solution of $\Omega(k_{1i}) = 0$.

where

$$F \equiv F_1 + F_2,\ F_j = F_{1i} + p_j F_{js}, \quad j = 1, 2.$$

The levels of per capita outputs of, the levels of capital intensities of, the levels of outputs of, the levels of capital employed by, the labor force employed by the two sectors are higher than the corresponding variable values of region 2. We also obtain that the consumption levels of commodity and services, the wage rate and the land rent in the advanced region are higher than in region 2. As the two regions are not greatly different in the land sizes, L_j, and amenity parameters, $\overline{\theta}_j$, and regional 1 is technologically more advanced than region 2, these results are expectable for a national economy with freely mobile production factors. The price of services in region 1 is lower than in region 2. This is mainly due to the fact that the productivity of the service sector of region 1 is higher than that of region 2. The advanced region's total output level, F_1, is much higher than that of region 2. Most of the population is concentrated in the advanced region.

The output, wealth, population, and wage income distribution between the two regions are given by

$$\begin{pmatrix}\hat{F}_1\\ \hat{F}_2\end{pmatrix} = \begin{pmatrix}0.75\\ 0.25\end{pmatrix}, \begin{pmatrix}\hat{K}_1\\ \hat{K}_2\end{pmatrix} = \begin{pmatrix}0.75\\ 0.25\end{pmatrix}, \begin{pmatrix}\hat{N}_1\\ \hat{N}_2\end{pmatrix} = \begin{pmatrix}0.714\\ 0.286\end{pmatrix}, \begin{pmatrix}\hat{W}_1\\ \hat{W}_2\end{pmatrix} = \begin{pmatrix}0.75\\ 0.25\end{pmatrix}, \tag{8.2.28}$$

where a variable x_j with circumflex, $\hat{x}_j$, denotes region j's share of the corresponding variable in the national economy. The shares of the output, capital stocks, population, and wage income of region 1 in the national economy are respectively 75 percent, 75 percent, 71.4 percent, and 75 percent. Most of the population and capital are located in region 1.

8.2.3 Comparative statics analysis

This section examines how one region may affect the national economy as its technology or amenity is improved. It is important to analyze how regional economic structures may be affected as the propensity or the total population to save is increased. As the system has a unique equilibrium, we can make comparative static analysis.

The effects of the productivity levels

First, we examine the case that all the parameters, except the productivity of region 1's product sector, A_{1i}, are the same as in (8.2.26). We increase the productivity level, A_{1i}, from 1.8 to 2. The simulation results are summarized in (8.2.30), in which a variable $\overline{\Delta} x_j$ stand for the change rate of the equilibrium value of the variable x_j

in percentage due to the change in the parameter value from A_{1i0} (= 1.8 in this case) to A_{1i} (= 2). That is

$$\overline{\Delta}x_j \equiv \frac{x_j(A_{1i}) - x_j(A_{1i0})}{x_j(A_{10})} \times 100,$$

where $x_j(A_{1i})$ stands for the equilibrium value of the variable x_j with the parameter value A_1 and $x_j(A_{1i0})$ stands for the value of the variable x_j with the parameter value A_{1i0}. We will use the symbol $\overline{\Delta}$ with the same meaning when we analyze other parameters.

We see that as A_{1i} is increased, the national output and wealth are increased and the rate of interest almost not affected. The per capita output levels of the two sectors in region 1 are increased, while the corresponding variables in the other region are not affected. As region 1's wage is increased, more people are attracted to region 1. As the per capita output levels of the two sectors and the population of region 1 are increased, the capita output levels of the two sectors are increased. As region 2's population is reduced and the region's per capita output levels of the two sectors are not affected, the output levels of the two sectors fall down. The lot size in region 1 is reduced and that of region 2 rises. The price of services rises in region 1 but is slightly changed in region 2. To examine how the service prices are affected, we look at the price determination equations in equations (8.2.3)

$$p_j = \frac{w_j}{\beta_{js} A_{js} k_{js}^{\alpha_{js}}}, \quad j = 1, 2. \tag{8.2.29}$$

We have

$$\overline{\Delta}p_j = \overline{\Delta}w_j - \alpha_{js}\overline{\Delta}k_{js}.$$

For region 1, the change in the price is positive as the change ratios in the wage rate and the capital intensity of the service sector are equal and α_{1s} is less than unity. For region 2, the change in the price is negligible as the changes in the wage rate and capital intensity of the service sector are negligible. It should be remarked that in this general equilibrium framework, the variable interactions are so complicated that it is difficult to explain any change in detail. Any change in a parameter will disturb the equilibrium and multiple forces interact with each other in disequilibrium before the new equilibrium point is achieved.

$$A_{1i}: 1.8 \Rightarrow 2.0 \Rightarrow \overline{\Delta}F = 13.94,\ \overline{\Delta}K = 13.94,\ \overline{\Delta}r = 0.0,$$

$$\begin{pmatrix} \overline{\Delta}f_{1i} \\ \overline{\Delta}f_{2i} \end{pmatrix} = \begin{pmatrix} 16.24 \\ 0 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}F_{1i} \\ \overline{\Delta}F_{2i} \end{pmatrix} = \begin{pmatrix} 25.06 \\ -18.97 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}f_{1s} \\ \overline{\Delta}f_{2s} \end{pmatrix} = \begin{pmatrix} 4.93 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta}F_{1s} \\ \overline{\Delta}F_{2s} \end{pmatrix} = \begin{pmatrix} 12.89 \\ -18.97 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}F_1 \\ \overline{\Delta}F_2 \end{pmatrix} = \begin{pmatrix} 25.06 \\ -18.97 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}k_{1i} \\ \overline{\Delta}k_{2i} \end{pmatrix} = \begin{pmatrix} 16.24 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}N_{1s}\\ \overline{\Delta}N_{2s}\end{pmatrix}=\begin{pmatrix}7.58\\ -18.97\end{pmatrix},\begin{pmatrix}\overline{\Delta}N_{1}\\ \overline{\Delta}N_{2}\end{pmatrix}=\begin{pmatrix}7.58\\ -18.97\end{pmatrix},\begin{pmatrix}\overline{\Delta}K_{1i}\\ \overline{\Delta}K_{2i}\end{pmatrix}=\begin{pmatrix}25.06\\ -18.97\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}K_{1s}\\ \overline{\Delta}K_{2s}\end{pmatrix}=\begin{pmatrix}25.06\\ -18.97\end{pmatrix},\begin{pmatrix}\overline{\Delta}k_{1s}\\ \overline{\Delta}k_{2s}\end{pmatrix}=\begin{pmatrix}16.24\\ 0\end{pmatrix},\begin{pmatrix}\overline{\Delta}N_{1i}\\ \overline{\Delta}N_{2i}\end{pmatrix}=\begin{pmatrix}7.58\\ -18.97\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}K_{1}\\ \overline{\Delta}K_{2}\end{pmatrix}=\begin{pmatrix}25.06\\ -18.97\end{pmatrix},\begin{pmatrix}\overline{\Delta}k_{1}\\ \overline{\Delta}k_{2}\end{pmatrix}=\begin{pmatrix}16.24\\ 0\end{pmatrix},\begin{pmatrix}\overline{\Delta}l_{1}\\ \overline{\Delta}l_{2}\end{pmatrix}=\begin{pmatrix}-7.05\\ 23.41\end{pmatrix}.$$

$$\begin{pmatrix}\overline{\Delta}c_{1i}\\ \overline{\Delta}c_{2i}\end{pmatrix}=\begin{pmatrix}16.24\\ 0\end{pmatrix},\begin{pmatrix}\overline{\Delta}c_{1s}\\ \overline{\Delta}c_{2s}\end{pmatrix}=\begin{pmatrix}4.93\\ 0\end{pmatrix},\begin{pmatrix}\overline{\Delta}w_{1}\\ \overline{\Delta}w_{2}\end{pmatrix}=\begin{pmatrix}16.24\\ 0\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}p_{1}\\ \overline{\Delta}p_{2}\end{pmatrix}=\begin{pmatrix}10.78\\ 0\end{pmatrix},\begin{pmatrix}\overline{\Delta}R_{1}\\ \overline{\Delta}R_{2}\end{pmatrix}=\begin{pmatrix}25.06\\ -18.97\end{pmatrix},\begin{pmatrix}\overline{\Delta}\hat{F}_{1}\\ \overline{\Delta}\hat{F}_{2}\end{pmatrix}=\begin{pmatrix}9.76\\ -28.88\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}\hat{K}_{1}\\ \overline{\Delta}\hat{K}_{2}\end{pmatrix}=\begin{pmatrix}9.76\\ -28.88\end{pmatrix}. \tag{8.2.30}$$

We summarize the effects of change in A_{1s} as in (8.2.31). The national output and total capital are increased as the productivity is improved. The output levels of the two sectors in region 1 are increased and the output levels of the two sectors in region 2 are reduced. More people migrate from region 2 to region 1. The wage rates are not affected. The price of services falls down as the productivity of the service sector is improved in region 1; the price of services is not changed in region 2. The land rent falls in region 2 and the land rent rises in region 1. A household in region 1 consumes more services and the same amount of consumption goods; a household in region 2 consumes the same amount of services and consumption goods.

$$A_{1s}: 1.5 \Rightarrow 1.7 \Rightarrow \overline{\Delta}F = 0.071, \overline{\Delta}K = 0.071, \overline{\Delta}r = 0,$$

$$\begin{pmatrix}\overline{\Delta}f_{1i}\\ \overline{\Delta}f_{2i}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},\begin{pmatrix}\overline{\Delta}F_{1i}\\ \overline{\Delta}F_{2i}\end{pmatrix}=\begin{pmatrix}0.61\\ -1.53\end{pmatrix},\begin{pmatrix}\overline{\Delta}f_{1s}\\ \overline{\Delta}f_{2s}\end{pmatrix}=\begin{pmatrix}13.33\\ 0\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}F_{1s}\\ \overline{\Delta}F_{2s}\end{pmatrix}=\begin{pmatrix}14.03\\ -1.53\end{pmatrix},\begin{pmatrix}\overline{\Delta}F_{1}\\ \overline{\Delta}F_{2}\end{pmatrix}=\begin{pmatrix}0.61\\ -1.53\end{pmatrix},\begin{pmatrix}\overline{\Delta}k_{1i}\\ \overline{\Delta}k_{2i}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}k_{1s}\\ \overline{\Delta}k_{2s}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},\begin{pmatrix}\overline{\Delta}N_{1i}\\ \overline{\Delta}N_{2i}\end{pmatrix}=\begin{pmatrix}0.61\\ -1.53\end{pmatrix},\begin{pmatrix}\overline{\Delta}N_{1s}\\ \overline{\Delta}N_{2s}\end{pmatrix}=\begin{pmatrix}0.61\\ -1.53\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}N_{1}\\ \overline{\Delta}N_{2}\end{pmatrix}=\begin{pmatrix}0.61\\ -1.53\end{pmatrix},\begin{pmatrix}\overline{\Delta}K_{1i}\\ \overline{\Delta}K_{2i}\end{pmatrix}=\begin{pmatrix}0.61\\ -1.53\end{pmatrix},\begin{pmatrix}\overline{\Delta}K_{1s}\\ \overline{\Delta}K_{2s}\end{pmatrix}=\begin{pmatrix}0.61\\ -1.53\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}K_{1}\\ \overline{\Delta}K_{2}\end{pmatrix}=\begin{pmatrix}0.61\\ -1.53\end{pmatrix},\begin{pmatrix}\overline{\Delta}k_{1}\\ \overline{\Delta}k_{2}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},\begin{pmatrix}\overline{\Delta}l_{1}\\ \overline{\Delta}l_{2}\end{pmatrix}=\begin{pmatrix}-0.61\\ 1.56\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta c_{1i}}\\ \overline{\Delta c_{2i}}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\overline{\Delta c_{1s}}\\ \overline{\Delta c_{2s}}\end{pmatrix}=\begin{pmatrix}13.33\\0\end{pmatrix},\begin{pmatrix}\overline{\Delta w_1}\\ \overline{\Delta w_2}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\overline{\Delta p_1}\\ \overline{\Delta p_2}\end{pmatrix}=\begin{pmatrix}-11.77\\0\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta R_1}\\ \overline{\Delta R_2}\end{pmatrix}=\begin{pmatrix}0.61\\-1.53\end{pmatrix},\begin{pmatrix}\overline{\Delta \hat{F}_1}\\ \overline{\Delta \hat{F}_2}\end{pmatrix}=\begin{pmatrix}0.54\\-1.60\end{pmatrix},\begin{pmatrix}\overline{\Delta \hat{K}_1}\\ \overline{\Delta \hat{K}_2}\end{pmatrix}=\begin{pmatrix}0.54\\-1.60\end{pmatrix}, \quad (8.2.31)$$

From the comparative statics analysis results in (8.2.30) and (8.2.31), we see that an improvement in region 1's productivity strongly affects the labor redistribution between the regions. This is particularly true with the productivity improvement in the industrial sector. An improvement in this sector's productivity strongly affects not only the redistribution of the population but also the regional shares of the output. It is also important to note that as a region improves its productivity, the other region's aggregated output levels fall not only in relative terms, but also in absolute values. This implies that if a region has a high rate of technological change and the other region remains technologically stationary, the economic activities tend to be shifted to in the technologically progressing region. In addition, we observe that as a region improves its technology, the other region's total output fall but all the individual households who remain in the region benefit in terms of utility. In this sense, a technological improvement benefits every individual in the economic system, but not every regional economy in aggregated terms. It can be seen that if we introduce some mechanism for endogenous technological changes, our analytical framework may be more powerful than contemporary theoretical approaches in explaining urban and regional agglomeration.[9]

Our analytical results might also be applied to provide some insights into the role of transportation in economic agglomeration. The new economic geography focuses on the interrelationship between increasing returns and transportation costs. Fujita *et al.* (1999) explain the approach. In those models, when the transportation costs are low, the economic activities tend to cluster due to a linkage effect. This approach is effective in explaining why as the impediments to the exchange of goods and the movement of people have decreased over a long enough period of time, many countries witnessed a massive concentration of economic activities within a fairly small number of urban regions. Although our model does not take account of transportation costs, our results show that economic activities are concentrated in technologically advanced regions. As reduction in transportation costs occurs for the two regions, we may interpret intuitively that a reduction in transportations means that productivity improvements occur in all the industrial sectors (whose products can be transported) in the economic system. As the technologically advanced region tends to benefit more from the productivity improvement, we see that reduction in transportation costs tend to lead to concentration of economic activities and people in technologically more advanced regions. Indeed, topics of endogenous technological change and transportation are too complicated to be studied in detail here.

We summarize the effects of change in A_{2s} as in (8.2.32). The national output and total capital are reduced as the productivity is improved. The output levels of

the two sectors in region 1 are increased and the output levels of the two sectors in region 2 are reduced. More people migrate from region 2 to region 1. The wage rates are not affected. The price of services falls down as the productivity of the service sector is improved in region 1; the price of services is not changed in region 2. The land rent falls in region 2 and the land rent rises in region 1. A household in region 1 consumes more services and the same amount of consumption goods; a household in region 2 consumes the same amount of services and consumption goods. It should be emphasized that as the productivity of region 2's service sector falls down, the national output falls down. As the productivity is improved, services become relatively cheaper. The region becomes more attractive for residents. More people will migrate from region 1 to region 2. As labor force is shifted to region 2, region 1's output falls down. As region 1's productivity is higher than region 2, the net result leads to reduction of the national output.

$$
\begin{aligned}
&A_{2s}: 1.3 \Rightarrow 1.5 \Rightarrow \overline{\Delta}F = -1.04, \overline{\Delta}K = -1.04, \overline{\Delta}r = 0,\\
&\begin{pmatrix}\overline{\Delta}f_{1i}\\ \overline{\Delta}f_{2i}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\overline{\Delta}F_{1i}\\ \overline{\Delta}F_{2i}\end{pmatrix} = \begin{pmatrix}-0.71\\1.77\end{pmatrix}, \begin{pmatrix}\overline{\Delta}f_{1s}\\ \overline{\Delta}f_{2s}\end{pmatrix} = \begin{pmatrix}0\\15.39\end{pmatrix},\\
&\begin{pmatrix}\overline{\Delta}F_{1s}\\ \overline{\Delta}F_{2s}\end{pmatrix} = \begin{pmatrix}-0.71\\1.77\end{pmatrix}, \begin{pmatrix}\overline{\Delta}F_{1}\\ \overline{\Delta}F_{2}\end{pmatrix} = \begin{pmatrix}-0.71\\-2.01\end{pmatrix}, \begin{pmatrix}\overline{\Delta}k_{1i}\\ \overline{\Delta}k_{2i}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix},\\
&\begin{pmatrix}\overline{\Delta}k_{1s}\\ \overline{\Delta}k_{2s}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\overline{\Delta}N_{1i}\\ \overline{\Delta}N_{2i}\end{pmatrix} = \begin{pmatrix}-0.71\\1.77\end{pmatrix}, \begin{pmatrix}\overline{\Delta}N_{1s}\\ \overline{\Delta}N_{2s}\end{pmatrix} = \begin{pmatrix}-0.71\\1.77\end{pmatrix},\\
&\begin{pmatrix}\overline{\Delta}N_{1}\\ \overline{\Delta}N_{2}\end{pmatrix} = \begin{pmatrix}-0.71\\1.77\end{pmatrix}, \begin{pmatrix}\overline{\Delta}K_{1i}\\ \overline{\Delta}K_{2i}\end{pmatrix} = \begin{pmatrix}-0.71\\1.77\end{pmatrix}, \begin{pmatrix}\overline{\Delta}K_{1s}\\ \overline{\Delta}K_{2s}\end{pmatrix} = \begin{pmatrix}-0.71\\1.77\end{pmatrix},\\
&\begin{pmatrix}\overline{\Delta}K_{1}\\ \overline{\Delta}K_{2}\end{pmatrix} = \begin{pmatrix}-0.71\\1.77\end{pmatrix}, \begin{pmatrix}\overline{\Delta}k_{1}\\ \overline{\Delta}k_{2}\end{pmatrix} = \begin{pmatrix}\overline{\Delta}c_{1i}\\ \overline{\Delta}c_{2i}\end{pmatrix} = \begin{pmatrix}\overline{\Delta}w_{1}\\ \overline{\Delta}w_{2}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix},\\
&\begin{pmatrix}\overline{\Delta}l_{1}\\ \overline{\Delta}l_{2}\end{pmatrix} = \begin{pmatrix}0.71\\-1.74\end{pmatrix}, \begin{pmatrix}\overline{\Delta}c_{1s}\\ \overline{\Delta}c_{2s}\end{pmatrix} = \begin{pmatrix}0\\15.39\end{pmatrix}, \begin{pmatrix}\overline{\Delta}p_{1}\\ \overline{\Delta}p_{2}\end{pmatrix} = \begin{pmatrix}0\\-13.33\end{pmatrix},\\
&\begin{pmatrix}\overline{\Delta}R_{1}\\ \overline{\Delta}R_{2}\end{pmatrix} = \begin{pmatrix}-0.71\\1.77\end{pmatrix}, \begin{pmatrix}\overline{\Delta}\hat{F}_{1}\\ \overline{\Delta}\hat{F}_{2}\end{pmatrix} = \begin{pmatrix}0.33\\-0.98\end{pmatrix}, \begin{pmatrix}\overline{\Delta}\hat{K}_{1}\\ \overline{\Delta}\hat{K}_{2}\end{pmatrix} = \begin{pmatrix}-0.63\\1.85\end{pmatrix}.
\end{aligned}
\tag{8.2.32}
$$

As shown in (8.2.33), as the productivity of region 2's industrial sector is increased, the national output is increased. We note that an improvement in the productivity of region 2's service sector reduces the national output, but an improvement in the productivity of the region's industrial sector increases the national output. As services are locally consumed and an increase in the supply of services improves the region's attractiveness as a residential area by reduction in the price, the improvement in the productivity of the region's

services sector will not benefit the national output. As the productivity of region 2's industrial sector is improved to the same level as the productivity of region 1, the labor force of the industrial sector shifted from region 1 to region 2 will not reduce the total output of the industrial sector of the two regions.

$$A_{2i}: 1.6 \Rightarrow 1.8 \Rightarrow \overline{\Delta}F = 4.09,$$

$$\begin{pmatrix} \overline{\Delta}F_{1i} \\ \overline{\Delta}F_{2i} \end{pmatrix} = \begin{pmatrix} -9.62 \\ 46.80 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}F_{1s} \\ \overline{\Delta}F_{2s} \end{pmatrix} = \begin{pmatrix} \overline{\Delta}N_1 \\ \overline{\Delta}N_2 \end{pmatrix} = \begin{pmatrix} -9.62 \\ 24.06 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}F_1 \\ \overline{\Delta}F_2 \end{pmatrix} = \begin{pmatrix} -9.62 \\ 44.66 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta}w_1 \\ \overline{\Delta}w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 18.32 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}p_1 \\ \overline{\Delta}p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 12.12 \end{pmatrix}. \tag{8.2.33}$$

The effects of the amenity parameters

We summarize the effects of change in the advanced region's amenity parameter, $\overline{\theta}_1$, in (8.2.33). As the advanced region's amenity parameter is increased, the national output is increased. As the parameter is increased, region 1 becomes more attractive as a residential area. Some people migrate from region 2 to region 1. More labor force is attracted to the region. As this region is technologically advanced, the migration will bring about increases in the national output, even though region 2's outputs are reduced. Region 1's output levels are increased as the region's amenity parameter is increased. The lot size of region 1 falls and the land rent of the region rises; the lot size of region 2 rises and the land rent falls. The per-capita consumption levels of consumption goods and services, the wage rates, and the prices of services are invariant as the amenity parameter is increased.

$$\overline{\theta}_1: 4 \Rightarrow 4.5 \Rightarrow \overline{\Delta}F = 0.79, \overline{\Delta}K = 0.79, \overline{\Delta}r = 0,$$

$$\begin{pmatrix} \overline{\Delta}f_{1i} \\ \overline{\Delta}f_{2i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}F_{1i} \\ \overline{\Delta}F_{2i} \end{pmatrix} = \begin{pmatrix} 6.83 \\ -17.08 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}f_{1s} \\ \overline{\Delta}f_{2s} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta}F_{1s} \\ \overline{\Delta}F_{2s} \end{pmatrix} = \begin{pmatrix} 6.83 \\ -17.08 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}F_1 \\ \overline{\Delta}F_2 \end{pmatrix} = \begin{pmatrix} 6.83 \\ -17.08 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}k_{1i} \\ \overline{\Delta}k_{2i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta}k_{1s} \\ \overline{\Delta}k_{2s} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}N_{1i} \\ \overline{\Delta}N_{2i} \end{pmatrix} = \begin{pmatrix} 6.83 \\ -17.08 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}N_{1s} \\ \overline{\Delta}N_{2s} \end{pmatrix} = \begin{pmatrix} 6.83 \\ -17.08 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta}N_1 \\ \overline{\Delta}N_2 \end{pmatrix} = \begin{pmatrix} 6.83 \\ -17.08 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}K_{1i} \\ \overline{\Delta}K_{2i} \end{pmatrix} = \begin{pmatrix} 6.83 \\ -17.08 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta}K_{1s} \\ \overline{\Delta}K_{2s} \end{pmatrix} = \begin{pmatrix} \overline{\Delta}K_1 \\ \overline{\Delta}K_2 \end{pmatrix} = \begin{pmatrix} 6.83 \\ -17.08 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}\,\overline{k}_1 \\ \overline{\Delta}\,\overline{k}_2 \end{pmatrix} = \begin{pmatrix} \overline{\Delta}c_{1i} \\ \overline{\Delta}c_{2i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}l_1\\ \overline{\Delta}l_2\end{pmatrix}=\begin{pmatrix}-6.39\\ 20.59\end{pmatrix},\ \begin{pmatrix}\overline{\Delta}c_{1s}\\ \overline{\Delta}c_{2s}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},\ \begin{pmatrix}\overline{\Delta}w_1\\ \overline{\Delta}w_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},\ \begin{pmatrix}\overline{\Delta}p_1\\ \overline{\Delta}p_2\end{pmatrix}=$$

$$\begin{pmatrix}\overline{\Delta}R_1\\ \overline{\Delta}R_2\end{pmatrix}=\begin{pmatrix}6.83\\ -17.08\end{pmatrix},\ \begin{pmatrix}\overline{\Delta}\hat{F}_1\\ \overline{\Delta}\hat{F}_2\end{pmatrix}=\begin{pmatrix}5.99\\ -17.73\end{pmatrix},\ \begin{pmatrix}\overline{\Delta}\hat{K}_1\\ \overline{\Delta}\hat{K}_2\end{pmatrix}=\begin{pmatrix}5.99\\ -17.73\end{pmatrix}$$

Although an increase in region 1's amenity increases the national an increase in region 1's amenity will not increase the national ou shown in (8.2.35), as region 2 improves its amenity, the national reduced.

$$\bar{\theta}_2: 3 \Rightarrow 3.5 \Rightarrow \overline{\Delta}F = -1.17.$$

$$\begin{pmatrix}\overline{\Delta}F_{1i}\\ \overline{\Delta}F_{2i}\end{pmatrix}=\begin{pmatrix}\overline{\Delta}F_{1s}\\ \overline{\Delta}F_{2s}\end{pmatrix}=\begin{pmatrix}\overline{\Delta}N_1\\ \overline{\Delta}N_2\end{pmatrix}=\begin{pmatrix}\overline{\Delta}F_1\\ \overline{\Delta}F_2\end{pmatrix}=\begin{pmatrix}-10.09\\ 25.25\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}w_1\\ \overline{\Delta}w_2\end{pmatrix}=\begin{pmatrix}\overline{\Delta}p_1\\ \overline{\Delta}p_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

To compare (8.2.34) and (8.2.35), we conclude that if amenity impr occurs in the technologically advanced region, then the national output if environmental improvement occurs in a technologically less advance the national output falls. This hints at the possibility that if the central go improves a technologically less advanced region, the national output This issue will be studied in future, as this study does not include any go intervention.

Another insight we obtain from the comparative statics analyses is ab disparities between the regions. Wage disparities may be caused by man such as spatial differences in the skill composition of the workforce, human endowments such (as climate and infrastructures), and in local inte Many empirical studies are made to study regional wage disparities in countries.[10] From our simulation result, we see that although wage affected by changes in regional differences technology, land and amenity, disparity is most strongly affected by change in technology. This also as far as neoclassical economy is concerned, if technological difference regions are not large, wage rates may tend to converge as the other factor affect the differences. It is interesting to note that from the empirical wage determination with a large panel of French workers, Combes *et a* have found that individual skills account for a large fraction of existin wage disparities with strong evidence of spatial sorting by skills and end only appear to play a small role. As far as relations between wage disp endowments are concerned, our model gives similar results. From (8.2 (8.2.35), we see that changes in amenity have negligible effects on wage

$$\begin{pmatrix}\overline{\Delta}l_1\\ \overline{\Delta}l_2\end{pmatrix}=\begin{pmatrix}-6.39\\ 20.59\end{pmatrix},\begin{pmatrix}\overline{\Delta}c_{1s}\\ \overline{\Delta}c_{2s}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},\begin{pmatrix}\overline{\Delta}w_1\\ \overline{\Delta}w_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},\begin{pmatrix}\overline{\Delta}p_1\\ \overline{\Delta}p_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}R_1\\ \overline{\Delta}R_2\end{pmatrix}=\begin{pmatrix}6.83\\ -17.08\end{pmatrix},\begin{pmatrix}\overline{\Delta}\hat{F}_1\\ \overline{\Delta}\hat{F}_2\end{pmatrix}=\begin{pmatrix}5.99\\ -17.73\end{pmatrix},\begin{pmatrix}\overline{\Delta}\hat{K}_1\\ \overline{\Delta}\hat{K}_2\end{pmatrix}=\begin{pmatrix}5.99\\ -17.73\end{pmatrix}. \tag{8.2.34}$$

Although an increase in region 1's amenity increases the national output, an increase in region 1's amenity will not increase the national output. As shown in (8.2.35), as region 2 improves its amenity, the national output is reduced.

$$\overline{\theta}_2: 3 \Rightarrow 3.5 \Rightarrow \overline{\Delta}F = -1.17,$$

$$\begin{pmatrix}\overline{\Delta}F_{1i}\\ \overline{\Delta}F_{2i}\end{pmatrix}=\begin{pmatrix}\overline{\Delta}F_{1s}\\ \overline{\Delta}F_{2s}\end{pmatrix}=\begin{pmatrix}\overline{\Delta}N_1\\ \overline{\Delta}N_2\end{pmatrix}=\begin{pmatrix}\overline{\Delta}F_1\\ \overline{\Delta}F_2\end{pmatrix}=\begin{pmatrix}-10.09\\ 25.25\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}w_1\\ \overline{\Delta}w_2\end{pmatrix}=\begin{pmatrix}\overline{\Delta}p_1\\ \overline{\Delta}p_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}. \tag{8.2.35}$$

To compare (8.2.34) and (8.2.35), we conclude that if amenity improvement occurs in the technologically advanced region, then the national output rises; but if environmental improvement occurs in a technologically less advanced region, the national output falls. This hints at the possibility that if the central government improves a technologically less advanced region, the national output may fall. This issue will be studied in future, as this study does not include any government intervention.

Another insight we obtain from the comparative statics analyses is about wage disparities between the regions. Wage disparities may be caused by many factors, such as spatial differences in the skill composition of the workforce, in non-human endowments such (as climate and infrastructures), and in local interactions. Many empirical studies are made to study regional wage disparities in different countries.[10] From our simulation result, we see that although wage rates are affected by changes in regional differences technology, land and amenity, the wage disparity is most strongly affected by change in technology. This also hints that as far as neoclassical economy is concerned, if technological differences among regions are not large, wage rates may tend to converge as the other factors weakly affect the differences. It is interesting to note that from the empirical study on wage determination with a large panel of French workers, Combes *et al.* (2003) have found that individual skills account for a large fraction of existing spatial wage disparities with strong evidence of spatial sorting by skills and endowments only appear to play a small role. As far as relations between wage disparity and endowments are concerned, our model gives similar results. From (8.2.34) and (8.2.35), we see that changes in amenity have negligible effects on wage disparity.

To further confirm this, we calculate the impact of the land sizes on the wage disparity

$$L_1 : 3 \Rightarrow 3.5 \Rightarrow \overline{\Delta}F = 0.08, \overline{\Delta}N_1 = 0.66, \overline{\Delta}N_2 = -1.65, \overline{\Delta}w_1 = \overline{\Delta}w_2 = 0;$$

$$L_2 : 4 \Rightarrow 4.5 \Rightarrow \overline{\Delta}F = -0.06, \overline{\Delta}N_1 = -0.51, \overline{\Delta}N_2 = 1.27, \overline{\Delta}w_1 = \overline{\Delta}w_2 = 0. \tag{8.2.36}$$

We see also that the land endowments have negligible effects on the wage disparity. As our conclusions are based on specified function forms, it remains to prove whether the observation by Combes *et al.* can be identified under more general conditions.

The effects of the preference change

We now examine change in the preference. As our results are based on the assumption of $\eta = 0.07$, we can change on γ, ξ, and λ. As $\eta + \xi + \lambda + \gamma = 1$, preference change should be subject to $\xi + \lambda + \gamma = 0.93$. We now reduce the propensity to save and increase the propensity to consume consumption goods as specified in (8.2.37). The effects of change in the nation's preference are provided in (8.2.37). As the propensity to save is reduced, the national output is decreased and the rate of interest rises. The per-capita output levels of all the sectors in the two regions are reduced. Some people migrate from the advanced region to the middle and poor regions. All the regions' output levels are reduced. The per-capita wealth levels and wage rates in all the regions are reduced. The per-capita consumption levels of consumption goods are increased. The land rents in the two regions fall. The share of national product of region 1 falls and the other region's share rises. There is no migration due to the specified preference change and the lot sizes are not changed. The per-capita consumption levels of services fall in the two regions. This is due to the specified pattern of the preference change. As the households increase the propensity to consume the consumption goods, the propensity to consume services relatively decline.

$$(\lambda, \xi) : (0.75, 0.1) \Rightarrow (0.7, 0.15) \Rightarrow$$

$$\overline{\Delta}F = -8.50, \overline{\Delta}K = -27.17, \overline{\Delta}r = 42.79,$$

$$\begin{pmatrix} \overline{\Delta}f_{1i} \\ \overline{\Delta}f_{2i} \end{pmatrix} = \begin{pmatrix} -9.97 \\ -9.97 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}F_{1i} \\ \overline{\Delta}F_{2i} \end{pmatrix} = \begin{pmatrix} -4.10 \\ -4.10 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}f_{1s} \\ \overline{\Delta}f_{2s} \end{pmatrix} = \begin{pmatrix} -9.54 \\ -9.54 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta}F_{1s} \\ \overline{\Delta}F_{2s} \end{pmatrix} = \begin{pmatrix} -22.45 \\ -14.28 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}F_{1} \\ \overline{\Delta}F_{2} \end{pmatrix} = \begin{pmatrix} -9.07 \\ -6.79 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}k_{1i} \\ \overline{\Delta}k_{2i} \end{pmatrix} = \begin{pmatrix} -26.89 \\ -26.89 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\Delta}k_{1s} \\ \overline{\Delta}k_{2s} \end{pmatrix} = \begin{pmatrix} -26.89 \\ -26.89 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}N_{1i} \\ \overline{\Delta}N_{2i} \end{pmatrix} = \begin{pmatrix} 5.35 \\ 5.35 \end{pmatrix}, \begin{pmatrix} \overline{\Delta}N_{1s} \\ \overline{\Delta}N_{2s} \end{pmatrix} = \begin{pmatrix} -14.28 \\ -14.28 \end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}N_1\\ \overline{\Delta}N_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\overline{\Delta}K_{1i}\\ \overline{\Delta}K_{2i}\end{pmatrix}=\begin{pmatrix}-22.99\\-22.99\end{pmatrix},\begin{pmatrix}\overline{\Delta}K_{1s}\\ \overline{\Delta}K_{2s}\end{pmatrix}=\begin{pmatrix}-37.33\\-37.33\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}K_1\\ \overline{\Delta}K_2\end{pmatrix}=\begin{pmatrix}-27.17\\-27.17\end{pmatrix},\begin{pmatrix}\overline{\Delta}\tilde{k}_1\\ \overline{\Delta}\tilde{k}_2\end{pmatrix}=\begin{pmatrix}-27.17\\-27.17\end{pmatrix},\begin{pmatrix}\overline{\Delta}l_1\\ \overline{\Delta}l_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}c_{1i}\\ \overline{\Delta}c_{2i}\end{pmatrix}=\begin{pmatrix}17.05\\17.05\end{pmatrix},\begin{pmatrix}\overline{\Delta}c_{1s}\\ \overline{\Delta}c_{2s}\end{pmatrix}=\begin{pmatrix}-22.45\\-22.45\end{pmatrix},\begin{pmatrix}\overline{\Delta}w_1\\ \overline{\Delta}w_2\end{pmatrix}=\begin{pmatrix}-8.97\\-8.97\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}p_1\\ \overline{\Delta}p_2\end{pmatrix}=\begin{pmatrix}0.63\\0.63\end{pmatrix},\begin{pmatrix}\overline{\Delta}R_1\\ \overline{\Delta}R_2\end{pmatrix}=\begin{pmatrix}-21.97\\-21.97\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}\hat{F}_1\\ \overline{\Delta}\hat{F}_2\end{pmatrix}=\begin{pmatrix}-0.63\\1.87\end{pmatrix},\begin{pmatrix}\overline{\Delta}\hat{K}_1\\ \overline{\Delta}\hat{K}_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}. \tag{8.2.37}$$

The conclusion that the labor distribution between the regions is invariant with regard to preference change is dependent on the specified pattern. For instance, if we allow the propensity to consume services to be changed and keep the propensity to consume consumption goods invariant as specified in (8.2.38), then the preference change affects labor distribution. As the propensity to consume services is increased, some people will migrate from region 2 to region 1. A main force for the direction is due to the assumption that the productivity of the service sector in region 1 is higher than in region 2. As the population is homogenous, people tend to move to the region which tends to offer services with lower price when the equilibrium is disturbed.

$$(\lambda,\gamma):(0.75,0.08)\Rightarrow(0.7,0.13)\Rightarrow\overline{\Delta}F=-8.07,\overline{\Delta}p_1=\overline{\Delta}p_2=0.66,$$

$$\begin{pmatrix}\overline{\Delta}N_1\\ \overline{\Delta}N_2\end{pmatrix}=\begin{pmatrix}0.09\\-0.22\end{pmatrix},\begin{pmatrix}\overline{\Delta}F_{1s}\\ \overline{\Delta}F_{2s}\end{pmatrix}=\begin{pmatrix}26.10\\35.59\end{pmatrix},$$

$$\begin{pmatrix}\overline{\Delta}c_{1i}\\ \overline{\Delta}c_{2i}\end{pmatrix}=\begin{pmatrix}-21.96\\-21.96\end{pmatrix},\begin{pmatrix}\overline{\Delta}c_{1s}\\ \overline{\Delta}c_{2s}\end{pmatrix}=\begin{pmatrix}25.99\\25.99\end{pmatrix}. \tag{8.2.38}$$

8.3 The dynamics of the monetary economy

This section examines the case of $\varepsilon_0 > 0$. We still specify the production functions as in (8.2.1). Hence, equations (8.2.2) and (8.2.3) still hold. We have $\phi_j(k_{1i}), \phi_r(k_{1i})$, $\phi_{js}(k_{1i}), \phi_{js}(k_{1i})$, and $\varphi_j(k_{1i})$ as in Section 8.2. By equations (8.1.3), (8.1.4) and (8.2.3), we have

$$\bar{y}_j=\overline{\phi}_j\left(k_{1i},\overline{k}_j,m\right)\equiv(1+\phi_r)\overline{k}_j+\phi_j+\mu m. \tag{8.3.1}$$

From the above equations and $(\pi + r)m_j = \varepsilon \bar{y}_j$ in (8.1.8), we have

$$\left(\frac{\pi + r}{\varepsilon}\right) m_j = (1 + \phi_r)\bar{k}_j + \phi_j + \mu m.$$

Multiplying the equation for region j in the above equations by N_j and then adding the two resulted equations, we have

$$\left(\frac{\pi + \phi_r}{\varepsilon} - \mu\right) Nm = (1 + \phi_r)K + \sum_{j=1}^{2} \phi_j N_j,$$

where we use equations (8.1.10) and (8.1.11). Insert $\pi = \mu - \dot{m}/m$ into the above equation

$$\dot{m} = (\mu + \phi_r - \varepsilon\mu)m - (1 + \phi_r)\frac{\varepsilon K}{N} - \frac{\varepsilon}{N}\sum_{j=1}^{2} \phi_j N_j. \tag{8.3.2}$$

From equations (8.1.12), we have

$$c_{js} N_j = N_{js} f_{js}, \quad j = 1, 2.$$

Substitute $p_j c_{js} = \gamma \bar{y}_j$ into the above equations

$$N_{js} = \frac{\gamma \bar{y}_j N_j}{p_j f_{js}}, \quad j = 1, 2. \tag{8.3.3}$$

From these equations and $N_{ji} + N_{js} = N_j$, we have

$$N_{ji} = \left(f_{js} - \frac{\gamma \bar{y}_j}{p_j}\right)\frac{N_j}{f_{js}}, \quad j = 1, 2. \tag{8.3.4}$$

Equations (8.3.3) and (8.3.4) determine the labor distribution between the two sectors as functions of the regional labor force, the disposable income, m and k_{1i} at each point of time.

In this section, we assume $a = 0$. We omit possible effects of the industrial sector on the region's amenity. Substitute equations (8.3.3) into equations (8.1.7)

$$\theta_j = \bar{\theta}_j \gamma^b p_j^{-b} \bar{y}_j^b N_j^{d+b}.$$

Substituting the above equations, $l_j = L_j/N_j$ and equations (8.1.8) into utility functions and applying $U_1 = U_2$, we obtain

$$n = p(k_{1i})\left(\frac{\bar{y}_1}{\bar{y}_2}\right)^{\bar{\eta}}, \tag{8.3.5}$$

in which

$$n \equiv \frac{N_2}{N_1},\ p(k_{1i}) \equiv \left(\frac{\overline{\theta}_1 L_1^{\eta_0}}{\overline{\theta}_2 L_2^{\eta_0}}\right)^{1/(d+b-\eta_0)} \left(\frac{p_2}{p_1}\right)^{(b+\gamma_0)/(d+b-\eta_0)},$$

$$\overline{\eta} \equiv \frac{b+\varepsilon_0+\xi_0+\gamma_0+\lambda_0}{d+b-\eta_0}.$$

From equation (8.1.11), we have

$$\sum_{j=1}^{2}\left(k_{ji}N_{ji}+k_{js}N_{js}\right)=\sum_{j=1}^{2}\overline{k}_j N_j. \tag{8.3.6}$$

Insert equations (8.3.3) and (8.3.4) in equation (8.3.6)

$$\sum_{j=1}^{2}\left(k_{ji}-\left(k_{ji}-k_{js}\right)\frac{\gamma\overline{y}_j}{p_j f_{js}}\right)N_j=\sum_{j=1}^{2}\overline{k}_j N_j. \tag{8.3.7}$$

We solve equation (8.3.7) in $\overline{k}_1$

$$\overline{k}_1=-n\overline{k}_2+\sum_{j=1}^{2}\left(k_{ji}-\left(k_{ji}-k_{js}\right)\frac{\gamma\overline{y}_j}{p_j f_{js}}\right)\frac{N_j}{N_1}. \tag{8.3.8}$$

By $\overline{y}_1=(1+\phi_r)\overline{k}_j+\phi_1+\mu m$, we obtain

$$\overline{k}_1=\frac{\overline{y}_1-\phi_1-\mu m}{1+\phi_r}. \tag{8.3.9}$$

Equal the right-hand sides of equations (8.3.8) and (8.3.9)

$$-n\overline{k}_2+\sum_{j=1}^{2}\left(k_{ji}-\left(k_{ji}-k_{js}\right)\frac{\gamma\overline{y}_j}{p_j f_{js}}\right)\frac{N_j}{N_1}=\frac{\overline{y}_1-\phi_1-\mu m}{1+\phi_r}.$$

Insert $n=p(k_{1i})(\overline{y}_1/\overline{y}_2)^{\overline{\eta}}$ in the above equation

$$\overline{y}_1-\overline{f}\left(k_{1i},\overline{k}_2,m\right)\overline{y}_1^{\overline{\eta}}-\hat{f}\left(k_{1i},\overline{k}_2,m\right)=0, \tag{8.3.10}$$

in which

$$\hat{f}\left(k_{1i},\overline{k}_2,m\right)\equiv\left(\frac{\phi_1+\mu m}{1+\phi_r}+k_{1i}\right)\frac{1}{\overline{f}_0\left(k_{1i},\overline{k}_2\right)},$$

$$\overline{f}\left(k_{1i},\overline{k}_2\right)\equiv\left(-\overline{k}_2+k_{2i}-(k_{2i}-k_{2s})\frac{\gamma\overline{y}_2}{p_2 f_{2s}}\right)\frac{p}{\overline{f}_0\left(k_{1i},\overline{k}_2\right)\overline{y}_2^{\overline{\eta}}},$$

$$\overline{f}_0(k_{1i})\equiv\frac{1}{1+\phi_r}+(k_{1i}-k_{1s})\frac{\gamma}{p_1 f_{1s}}.$$

By $\phi_r + \delta_{k1} = \alpha_{1s} p_1 f_{1s}/k_{1s}$ and the definition of $\bar{f}_0(k_{1i})$, we have

$$\bar{f}_0(k_{1i}) = \frac{(1-\alpha_{1s}\gamma)\phi_r + \delta_{k1} - \alpha_{1s}\gamma}{(1+\phi_r)(\phi_r+\delta_{k1})} + \frac{\alpha_{1s}\gamma k_{1i}}{(\phi_r+\delta_{k1})k_{1s}}.$$

As in Section 8.2, we assume $\bar{f}_0(k_{1i}) > 0$. As $\bar{f}_0(k_{1i}) > 0$, we have $\hat{f} > 0$. It is impossible to explicitly solve equation (8.3.10). We assume that equation (8.3.10) has at least one positive solution, denoted as

$$\bar{y}_1 = \Lambda_0\left(k_{1i}, \bar{k}_2, m\right). \tag{8.3.11}$$

Substitute (8.3.11) into equation (8.3.9)

$$\bar{k}_1 = \Lambda\left(k_{1i}, \bar{k}_2, m\right) \equiv \frac{\Lambda_0\left(k_{1i}, \bar{k}_2, m\right) - \phi_1(k_{1i}) - \mu m}{1+\phi_r(k_{1i})}. \tag{8.3.12}$$

From equations (8.3.11) and (8.3.1) and $n = p(k_{1i})(\bar{y}_1/\bar{y}_2)^{\bar{\eta}}$, we see that n is a function of k_{1i}, $\bar{k}_2$, and m. From $N_1 + N_2 = N$ and $n = N_2/N_1$, we solve

$$N_1 = \frac{N}{1+n},\ N_2 = \frac{nN}{1+n}. \tag{8.3.13}$$

We see that N_1 and N_2 are functions of k_{1i}, $\bar{k}_2$, and m. From $K = \sum_j \bar{k}_j N_j$, K is a function of k_{1i}, $\bar{k}_2$, and m. We see that the right-hand side of equation (8.3.2) is a function of k_{1i}, $\bar{k}_2$, and m. That is

$$\dot{m} = \Omega_m\left(k_{1i}, \bar{k}_2, m\right). \tag{8.3.14}$$

From $(\pi + r)m_j = \varepsilon \bar{y}_j$ and $\pi = \mu - \dot{m}/m$, we have

$$m_j = \bar{m}_j\left(k_{1i}, \bar{k}_2, m\right) \equiv \frac{\varepsilon \bar{y}_j}{\mu - \Omega_m/m + \phi_r}. \tag{8.3.15}$$

Taking derivatives of equations (8.3.15) with respect to t yields

$$\dot{m}_j = \frac{\partial \bar{m}_j}{\partial k_{1i}}\dot{k}_{1i} + \frac{\partial \bar{m}_j}{\partial \bar{k}_2}\dot{\bar{k}}_2 + \Omega_m \frac{\partial \bar{m}_j}{\partial k_{1i}}, \quad j = 1, 2. \tag{8.3.16}$$

Substitute $s_j = \lambda \bar{y}_j$ from equations (8.1.8) into equations (8.1.9)

$$\dot{a}_j = \lambda \bar{y}_j - a_j.$$

Insert $a_j = \overline{k}_j + m_j$ and equations (8.3.1) and (8.3.16) into the above equations

$$\dot{\overline{k}}_1 = \lambda\Lambda_0 - \Lambda_1 - \overline{m}_1 + \Omega_m \frac{\partial \overline{m}_1}{\partial k_{1i}} - \frac{\partial \overline{m}_1}{\partial k_{1i}} \dot{k}_{1i} + \frac{\partial \overline{m}_1}{\partial \overline{k}_2} \dot{\overline{k}}_2,$$

$$\dot{\overline{k}}_2 = \lambda\overline{\phi}_2 - \overline{k}_2 - \overline{m}_2 - \frac{\partial \overline{m}_2}{\partial k_{1i}} \dot{k}_{1i} + \frac{\partial \overline{m}_2}{\partial \overline{k}_2} \dot{\overline{k}}_2 + \Omega_m \frac{\partial \overline{m}_2}{\partial k_{1i}}, \tag{8.3.17}$$

where we also use equation (8.3.12). Taking derivatives of equation (8.3.12) with respect to t yields

$$\dot{\overline{k}}_1 = \frac{\partial \Lambda}{\partial k_{1i}} \dot{k}_{1i} + \frac{\partial \Lambda}{\partial \overline{k}_2} \dot{\overline{k}}_2 + \Omega_m \frac{\partial \Lambda}{\partial m}. \tag{8.3.18}$$

We do not explicitly calculate partial derivatives as it is straightforward but the expressions are tedious. Equating both right-hand sides of equation (8.2.18) and the first equation in (8.2.17), we solve

$$\frac{\partial \Lambda}{\partial k_{1i}} \dot{k}_{1i} + \frac{\partial \Lambda}{\partial \overline{k}_2} \dot{\overline{k}}_2 + \Omega_m \frac{\partial \Lambda}{\partial m} = \lambda\Lambda_0 - \Lambda_1 - \overline{m}_1 + \Omega_m \frac{\partial \overline{m}_1}{\partial k_{1i}} - \frac{\partial \overline{m}_1}{\partial k_{1i}} \dot{k}_{1i} + \frac{\partial \overline{m}_1}{\partial \overline{k}_2} \dot{\overline{k}}_2.$$

We now express this equation and the second equation in matrix form with $\dot{k}_{1i}$ and $\dot{\overline{k}}_2$ as follows

$$H_0\left(k_{1i}, \overline{k}_2, m\right) \begin{bmatrix} \dot{k}_{1i} \\ \dot{\overline{k}}_2 \end{bmatrix} = \begin{bmatrix} h_1\left(k_{1i}, \overline{k}_2, m\right) \\ h_2\left(k_{1i}, \overline{k}_2, m\right) \end{bmatrix}, \tag{8.3.19}$$

where

$$H_0\left(k_{1i}, \overline{k}_2, m\right) \equiv \begin{bmatrix} \dfrac{\partial \overline{m}_2}{\partial k_{1i}} & 1 - \dfrac{\partial \overline{m}_2}{\partial \overline{k}_2} \\ \dfrac{\partial \Lambda}{\partial k_{1i}} + \dfrac{\partial \overline{m}_1}{\partial k_{1i}} & \dfrac{\partial \Lambda}{\partial \overline{k}_2} - \dfrac{\partial \overline{m}_1}{\partial \overline{k}_2} \end{bmatrix},$$

$$h_1\left(k_{1i}, \overline{k}_2, m\right) \equiv \lambda\overline{\phi}_2 - \overline{k}_2 - \overline{m}_2 + \Omega_m \frac{\partial \overline{m}_2}{\partial k_{1i}},$$

$$h_2\left(k_{1i}, \overline{k}_2, m\right) \equiv \lambda\Lambda_0 - \Lambda_1 - \overline{m}_1 + \Omega_m \frac{\partial \overline{m}_1}{\partial k_{1i}} - \Omega_m \frac{\partial \Lambda}{\partial m}$$

If the inverse of the matrix $H_0\left(k_{1i}, \overline{k}_2, m\right)$ exists, we solve linear equations as follows

$$\dot{\overline{k}}_{1i} = H_1\left(k_{1i}, \overline{k}_2, m\right),$$

$$\dot{\overline{k}}_2 = H_2\left(k_{1i}, \overline{k}_2, m\right),$$

where we can calculate H_j from equations (8.3.19).

Lemma 8.3.1
The dynamics of the national economy is given by the following 3-dimensional differential equations with $k_{1i}(t)$, $\overline{k}_2(t)$, and $m(t)$ as the variables

$$\dot{\overline{k}}_{1i} = H_1\left(k_{1i}\overline{k}_2, m\right),$$

$$\dot{\overline{k}}_2 = H_2\left(k_{1i}, \overline{k}_2, m\right),$$

$$\dot{m} = \Omega_m\left(k_{1i}, \overline{k}_2, m\right).$$

For any given positive values of $k_{1i}(t)$, $\overline{k}_2(t)$ and $m(t)$ at any point of time, all the other variables are determined by the following procedure: r, p_j, w_j, k_{2i}, k_{js}, $j = 1, 2$ by (8.2.2) and (8.2.3) $\rightarrow$ m_j by (8.3.15) $\rightarrow$ $\overline{k}_1$ by (8.3.12) $\rightarrow \overline{y}_1$ by (8.3.11) $\rightarrow \overline{y}_2$ by (8.3.1) $\rightarrow \pi = \mu - \dot{m}/m \rightarrow f_{jq} = A_{jq}k_{jq}^{\alpha_{jq}}$, $q = i, s \rightarrow n = p(k_{1i})(\overline{y}_1/\overline{y}_2)^{\overline{\eta}}$ $\rightarrow$ $N_j, j = 1, 2$, by (8.3.13) $\rightarrow l_j = L_j/N_j \rightarrow R_j = \eta\overline{y}_j/l_j \rightarrow c_{ji}$, c_{js}, and s_j by (8.1.8) $\rightarrow$ N_{js} by (8.3.3) $\rightarrow$ N_{ji} by (8.3.4) $\rightarrow$ $F_{jq} = N_{jq}f_{jq}$.

Following Lemma 8.3.1, we can determine all the variables in the national economic system with proper initial conditions. We may examine the dynamics of the system as in Section 8.2. As few new insights can be obtained and simulation is straightforward (as Lemma 8.3.1 provides the procedure), we will not further examine behavior of the system.

8.4 Conclusions

This chapter proposed a two-region two-sector monetary growth model with capital accumulation and endogenous amenity under assumptions of profit maximization, utility maximization, and perfect competition. We used the utility function, which determines saving and consumption with utility optimization without leading to a higher dimensional dynamic system like, for example, the Ramsey approach. The dynamics of 2-region non-monetary national economy is controlled by a 2-dimensional differential equations system. We also simulated the non-monetary economy with the Cobb-Douglas production functions and demonstrated the existence of a unique equilibrium point and effects of changes in some parameters. Our comparative statics analysis provides some important insights. For instance,

if environmental improvement occurs in the technologically advanced region, the national output rises; but if environmental improvement occurs in a technologically less advanced region, the national output falls. This hints at the possibility that if the central government improves a technologically less advanced region, the national output may fall. With regard to technological change, we observe that as a region improves its technology, its land rent increases and the land rent in the other region falls. As a region improves its technology, the other region's aggregated output level fall not only in relative terms, but also in absolute values. This implies that if any region has a high rate of technological change and the other region remains technologically stationary, the economic activities and the labor force tend to be concentrated in the technologically advancing region. Also we observe that as a region improves its technology, the other region's total output falls but all the households in the other region benefit in terms of consumption of goods, housing and wealth. Although the population is homogeneous, the impact of its preference change on the regions is not symmetry. For instance, under certain conditions, as the propensity to save is reduced, the per-capita output levels of all the regions are reduced. It should be remarked that our model demonstrates divergence of wages rates among regions in the long term. From our simulation result, we see that although wage rates are affected by changes in regional differences in technology, land and amenity, the wage disparity is mostly strongly affected by change in regional technology. This also hints that as far as neoclassical economy is concerned, if technological differences among regions are not large, wage rates may tend to converge as the other factors weakly affect the differences. Finally, we provide the procedure to determine all the variables in the monetary economy. We do not simulate the model partly because we already provided the procedure for computing and partly few new insights may be obtained.

It is straightforward to introduce the CIA approach and other approaches, such as money in production function approach, to the interregional dynamics proposed in this chapter. As the model is structurally general, it is possible to deal with various national as well as regional growth and environment issues. Given the contemporary literature on (spaceless) economic growth and the modeling framework proposed in this study, we may extend and generalize the model in different ways. We may analyze behavior of the model with other forms of production or utility functions. There are multiple production sectors and households are not homogenous. Any extension will cause some analytical difficulties because of the nature of regional dynamics. Moreover, it is known that issues related to tax competition among regions has increasingly caused great interests in economic geography.[11] We can extend the dynamic equilibrium framework proposed in this study to deal with these issues. Another important issue is seldom analyzed in interregional economic theory. That is, how preference patterns will interact with regional economic structures and development. There are different goods, such as inferior goods, normal goods, and luxury goods, whose propensities are changeable as income and wealth are changed. In addition, propensity to save should vary as economic conditions are changed. Nevertheless, the contemporary economic growth theory does not

provide effective analytical frameworks to deal with these important issues. As demonstrated by my works,[12] the framework used in this chapter can effectively take account of dynamic interactions between preference changes and economic development. It is important to examine these issues in economic geography.

APPENDIX

A.8.1 The dynamics with private ownership of land and general production functions

We now show a procedure to determine the dynamics of the system in two differential equations with general production functions. Here we discuss a case that each household owns L_j/N amount of land in region j and it is impossible to sell land but it is free to rent one's own land to others. Under this assumption, the land revenue $\bar{r}$ per worker is given as follows

$$\bar{r}(t) = \frac{1}{N}\sum_{j=1}^{J} L_j R_j.$$

Hence, each worker receives the land revenue $\bar{r}$ irrespective of his dwelling region. In this case, equation (8.1.3) is changed as follows

$$y_j(t) = \bar{r}(t) + r(t)\bar{k}_j(t) + w_j(t), \quad j = 1, 2. \tag{8.1.3}$$

It should be noted that $y_j(t)$ in equations (8.1.4) and (8.1.5) should be replaced by equations (8.1.3) in the appendix.

First, from equations (8.1.2) we obtain

$$f'_{2i}(k_{2i}) = f'_{1i}(k_{1i}) - \delta, \tag{A.8.1.1}$$

where $\delta \equiv \delta_{k1} - \delta_{k2}$. Equation (A.8.1.1) determines a unique relation between k_{1i} and k_{2i}, denoted by

$$k_{2i} = \phi_{2i}(k_{1i}), \tag{A.8.1.2}$$

where $\phi_1(k_1) = k_1$. From equations (A.8.1.1), we have

$$f''_{2i}(k_{2i})\frac{dk_{2i}}{dk_{1i}} = f''_{1i}(k_{1i}).$$

As $f''_{ji}(k_{ji}) \leq 0$, $j = 1, 2$, we see that $dk_{2i}/dk_{1i} \geq 0$. That is,

$$\phi'_{2i}(k_{1i}) \geq 0.$$

Hence, for any given $k_{1i}(t) > 0$, we determine $k_{2i}(t)$ as a unique function of $k_{1i}(t)$. From equations (8.1.2), we determine the wage rates and the rate of interest as functions of $k_{1i}(t)$ as follows

$$r = \phi(k_{1i}) \equiv f'_{1i}(k_{1i}) - \delta_{k1}, w_j = \overline{\phi}_j(k_{1i}) \equiv f_{ji}\left(\phi_{ji}(k_{1i})\right) - \phi_{ji}(k_{1i}) f'_{ji}\left(\phi_{ji}(k_{1i})\right), \tag{A.8.1.3}$$

in which $\phi_{1i} = k_{1i}$. From equations (8.1.2), we also obtain

$$\frac{f_{js}\left(k_{js}\right) - k_{js} f'_{js}\left(k_{js}\right)}{f'_{js}\left(k_{js}\right)} = \frac{\overline{\phi}_j(k_{1i})}{\phi(k_{1i})}. \tag{A.8.1.4}$$

Equations (A.8.1.4) determine k_{js} as functions of $k_{1i}(t)$ as follows

$$k_{js} = \phi_{js}(k_{1i}).$$

From equations (8.1.2), we obtain

$$p_j = \frac{\phi(k_{1i}) + \delta_{kj}}{f'_{js}\left(\phi_{js}(k_{1i})\right)}. \tag{A.8.1.5}$$

The services' prices can also be expressed as functions of $k_{1i}(t)$.

From equations (8.1.12), we have

$$c_{js} N_j = N_{js} f_{js}, \quad j = 1, 2.$$

In this section, we always require $\eta_0 + \gamma_0 + \xi_0 + \lambda_0 = 1$. We have $\eta = \eta_0$, $\gamma = \gamma_0$, $\xi = \xi_0$, and $\lambda = \lambda_0$. Substitute $c_{js} = \gamma p_j \hat{y}_j$ into the above equation

$$N_{js} = \frac{\gamma p_j \hat{y}_j N_j}{f_{js}}, \quad j = 1, 2. \tag{A.8.1.6}$$

From this equation and $N_{ji} + N_{js} = N_j$, we have

$$N_{ji} = \left(f_{js} - \gamma p_j \hat{y}_j\right) \frac{N_j}{f_{js}}, \quad j = 1, 2. \tag{A.8.1.7}$$

Equations (A.8.1.6) and (A.8.1.7) determine the labor distribution between the two sectors as functions of the regional labor force, the disposable income, and k_{1i} at each point of time.

By $l_j R_j = \eta \hat{y}_j$, $l_j N_j = L_j$, and the definition of $\bar{r}$, we have

$$\bar{r} = \frac{\eta(\hat{y}_1 N_1 + \hat{y}_2 N_2)}{N} = \frac{\eta(\hat{y}_1 + \hat{y}_2 n)}{1 + n}, \tag{A.8.1.8}$$

where $n \equiv N_2/N_1$. Substitute equation (A.8.1.8) into equations (8.1.5)

$$\hat{y}_j = \frac{\eta(\hat{y}_1 + \hat{y}_2 n)}{1+n} + (1+r)\bar{k}_j + w_j, \quad j = 1, 2.$$

We solve the above equations

$$\hat{y}_1 = \left[ng_0(n)\left((1+r)\bar{k}_2 + w_2\right) + (1+r)\bar{k}_1 + w_1\right]\frac{g(n)}{g(n) - \eta(1+n)},$$

$$\hat{y}_2 = \left[g_0(n)\left((1+r)\bar{k}_1 + w_1\right) + (1+r)\bar{k}_2 + w_2\right]\frac{(1+n)(1+n-\eta)}{g(n) - \eta(1+n)}.$$

in which

$$g_0(n) \equiv \frac{\eta}{1+n-\eta n}, \quad g(n) \equiv (1+n)(1+n-\eta n).$$

Substitute equations (8.1.7) and (8.1.8), and $l_j = L_j/N_j$ into utility functions (8.1.6)

$$U_j = \bar{\theta}_j \xi^{\xi} \gamma^{\gamma} \lambda^{\lambda} L_j^{\eta} p_j^{\gamma} f_{ji}^{a} f_{js}^{b} N_{ji}^{a} N_{js}^{b} N_j^{d-\eta} \hat{y}_j^{\xi+\gamma+\lambda}.$$

Insert equations (A.8.1.6) and (A.8.1.7) into the above utility functions

$$U_j = \bar{\theta}_j \xi^{\xi} \gamma^{b+\gamma} \lambda^{\lambda} L_j^{\eta} p_j^{b+\gamma} f_{ji}^{a} f_{js}^{-a} \left(f_{js} - \gamma p_j \hat{y}_j\right)^{a} N_j^{d+a+b-\eta} \hat{y}_j^{b+\xi+\gamma+\lambda}.$$

Apply $U_1 = U_2$ to the above formulas

$$n = \Lambda_0\left(\bar{k}_1, \bar{k}_2, k_{1i}, n\right)$$
$$\equiv \left[\frac{\bar{\theta}_1 L_1^{\eta}}{\bar{\theta}_2 L_2^{\eta}}\left(\frac{p_1}{p_2}\right)^{b+\gamma}\left(\frac{f_{1i} f_{2s}}{f_{1s} f_{2i}}\right)^{a}\left(\frac{f_{1s} - \gamma p_1 \hat{y}_1}{f_{2s} - \gamma p_2 \hat{y}_2}\right)^{a}\left(\frac{\hat{y}_1}{\hat{y}_2}\right)^{b+\xi+\gamma+\lambda}\right]^{1/(d+a+b-\eta)}. \tag{A.8.1.9}$$

In general, we cannot explicitly solve the function $n = \Lambda_0\left(\bar{k}_1, \bar{k}_2, k_{1i}, n\right)$ in n. Let us assume that from equation (A.8.1.9) we can express n as a function $\bar{k}_1, \bar{k}_2$, and k_{1i} as follows

$$n = \Lambda\left(\bar{k}_1, \bar{k}_2, k_{1i}\right).$$

From $n = N_2/N_1$ and $N_1 + N_2 = N$, we solve

$$N_1 = \Lambda_1\left(\bar{k}_1, \bar{k}_2, k_{1i}\right) \equiv \frac{N}{1+\Lambda}, \quad N_2 = \Lambda_2\left(\bar{k}_1, \bar{k}_2, k_{1i}\right) \equiv \frac{\Lambda N}{1+\Lambda}.$$

From equation (8.1.11), we have

$$\sum_{j=1}^{2}\left(k_{ji}N_{ji}+k_{js}N_{js}\right)=\sum_{j=1}^{2}\overline{k}_{j}N_{j}. \tag{A.8.1.10}$$

It is not difficult to see that this equation contains only three variables $\overline{k}_1, \overline{k}_2$, and k_{1i}. Assume that we can solve (A.8.1.10) as follows

$$\overline{k}_1=\Psi\left(\overline{k}_1,k_{1i}\right).$$

Substitute $s_j=\lambda\hat{y}_j$ from equations (8.1.8) into equations (8.1.9)

$$\dot{\overline{k}}_j=\lambda\hat{y}_j-\overline{k}_j.$$

From the above equations and the above analyses, we obtain

$$\dot{\overline{k}}_j=\Lambda_j\left(k_{1i},\overline{k}_1,\overline{k}_2\right),\quad j=1,2.$$

Taking derivatives of equation (A.8.1.10) with respect to t yields

$$\dot{\overline{k}}_1=\frac{\partial\Psi}{\partial k_{1i}}\dot{k}_{1i}+\Lambda_2\frac{\partial\Lambda}{\partial\overline{k}_2}. \tag{A.8.1.11}$$

From equations (A.8.1.5) and (A.8.1.11), we describe the dynamic system of the whole system by the following differential equations

$$\dot{\overline{k}}_{1i}=\widetilde{\Lambda}_1\left(k_{1i}\overline{k}_2\right),$$

$$\dot{\overline{k}}_2=\widetilde{\Lambda}_2\left(k_{1i}\overline{k}_2\right).$$

9 Money, growth, and international trade

The previous chapters focused on monetary growth models for national economies with capital accumulation, knowledge growth, heterogeneous households, economic structures, urban structures and interregional trade. This chapter extends the national economic models to international economies. Although extensions are conceptually not difficult, it is sometimes difficult to analyze the behavior of trade models. This chapter will study a few models with international trade. As monetary trade issues are so complicated, we will only represent a few simple models.

9.1 A classical monetary trade model

We now introduce a classical model of monetary open economy with the gold standard.[1] This model and its various extensions provide insights into many important monetary issues.[2] The model deals with the interactions among money, spending and prices in the open economy with full price flexibility. Spending is linked to money (gold) holdings. The world consists of two economies, called domestic and foreign countries. There is only one good in the world, the price of which is spatially arbitraged. Let a variable with tilde stand for foreign country. Let G stand for the world stock of gold and M and $\tilde{M}$ the money stocks, all measured in a common form, for example yen. The money is the only asset. The world money stock satisfies

$$G = M + \tilde{M}. \tag{9.1.1}$$

Assume that the link between money and spending is represented by the following form of a constant expenditure velocity

$$C = VM, \tilde{C} = \tilde{V}\tilde{M}, \tag{9.1.2}$$

Where C and $\tilde{C}$ are national nominal spending levels in the domestic and foreign countries and V and $\tilde{V}$ are the constant expenditure velocities.

Let the two economies supply fixed amount of output, Y and $\tilde{Y}$. Let P stand for the globally equal price of the good. The balance of world income and spending is given by

$$\left(Y+\tilde{Y}\right)P = VM + \tilde{V}\tilde{M}.$$

In the reminder of this section, we require $V = \tilde{V}$. The model is closed by the following equation for the balance of payments and money supply

$$\dot{M} = PY - VM = -\dot{\tilde{M}}, \tag{9.1.3}$$

The equation says that the balance of payments surplus is equal to the trade surplus, which is equal to the excess of demand over supply. Substituting $P = VG/\left(Y+\tilde{Y}\right)$ into equation (9.1.3) yields

$$\dot{M} = (yG - M)V, \tag{9.1.4}$$

where $y \equiv Y/(Y+\tilde{Y})$. We see that a country's economic growth will improve its balance of payments. Evidently, the differential equation has a unique stable equilibrium point. This equation is the central equation of classical international monetary economics. The equation implies that payments imbalances redistribute money over time until the world economy converges to the natural distribution. At equilibrium, the share of the domestic economy in the world money supply depends on the economy's output share, y.

The simple model can also be used to discuss other issues. For instance, we may allow velocities to vary among countries. An important extension is to discuss the case that countries produce non-trade goods. We are still concerned with a world economy with identical velocities and a single trade good. We now assume that each country produces a fixed amount of non-trade good per unit of time. The question is to determine the price of the traded good and the balance of payments as well as the prices of the non-traded goods. Let subscripts T and N denote respectively the traded and non-traded goods sectors. Equilibrium in the market for non-traded goods requires

$$P_N Y_N = \beta V M, \tag{9.1.5}$$

where β is the given expenditure share of domestic goods. Equilibrium condition in the market for traded goods is

$$\left(Y_N + \overline{Y}_N\right)P_T = \beta V G. \tag{9.1.6}$$

The balance of payments, which is equal to the excess of income over spending when goods market clears, is represented by

$$\dot{M} = P_T Y_T - (1-\beta)VM.$$

From the above equation and equations (9.1.5) and (9.1.6), we obtain

$$\dot{M} = (1-\beta)V(y_T G - M), \tag{9.1.7}$$

where $y_T \equiv Y_T/(Y_T + \tilde{Y}_T)$. The equation reveals that the balance of payments is affected by the presence of non-traded goods, measured by β. The higher the value of β, the more gradually the specie flow mechanism operates. It should be noted that in the long run $y_T G = M$, which is not affected by β. From equations (9.1.5) and (9.1.6), we obtain

$$p = \frac{M}{y_N G}, \tag{9.1.8}$$

where $p \equiv P_N/P_T$ and $y_N \equiv Y_N/(Y_N + \tilde{Y}_N)$. As y_N and G are fixed, we see that the relative price moves positively in proportion to M.

9.2 Dornbush's exchange rate model for a small economy

In this section we are concerned with a single country. The rest of world is also treated as a single economy with a single currency. When we use the term the "exchange rate," it is then the price of the single foreign currency in terms of the currency of the country under consideration. For instance, when we refer to the foreign currency as dollars and the domestic currency as Swedish krona, the exchange rate is then the price of dollars in kronor. If 1 US dollar is exchanged for 7 Swedish kronor, then the exchange rate is 7. It is well known that Mundell (1960, 1965) and Fleming (1962) introduced the IS-LM analysis of a closed economy to an open one.[3] The model assumes that international capital mobility is perfect, meaning that home and foreign bonds are perfect substitutes. Hence, interest rates are equalized by incipient international reserve flows. The model is given by three equations

$$\frac{M}{P} = L(r,y),$$

$$y = A(r,y) + Ex\left(y, \tilde{y}, \frac{e\tilde{P}}{P}\right),$$

$$r = \tilde{r},$$

where r and $\tilde{r}$ are respectively the home and foreign interest rates, Ex denotes the net exports trade balance and $e\tilde{P}/P$ is the real exchange rate which influences net exports. It should be noted that as the foreign interest rate is fixed, the movement of capital is ultimately a function of the domestic interest rate. We can therefore rewrite the following equation for the equilibrium in the balance of payment

$$Ex\left(y, \tilde{y}, \frac{e\tilde{P}}{P}\right) + K(r, r^*) = 0,$$

where $K > (<) 0$ is the net capital inflow (outflow). In this model, the world interest rate, foreign price and foreign income are given. Under fixed exchange rate the exchange rate is given, but money stock is endogenous; under flexible exchange rates the money stock is determined by the central government. This model has played the role of the key model in the literature of small open economies. As it is straightforward to analyze the model and its implications are explored in standard textbooks, we will not study it further.[4] In Appendix A.9.1, we introduce the standard IS-LM model with international trade.

Another well-known small country trade model is the so-called Dornbusch exchange-rate model.[5] The model is described as follows

$$y^d(t) = \delta(e(t) + \tilde{p} - p(t)) - \sigma(r(t) - p(t+1) + p(t)), \tag{9.2.1}$$

$$p(t+1) - p(t) = \alpha\left(y^d(t) - y\right), \tag{9.2.2}$$

$$m - p(t) = \phi y - \lambda r(t), \tag{9.2.3}$$

$$r(t) = \tilde{r} + e(t+1) - e(t), \tag{9.2.4}$$

where δ, $\tilde{p}$, σ, α, y, m, ϕ, λ, and $\tilde{r}$ are positive parameters, $y^d(t)$ and $p(t)$ are logarithms of domestic aggregate demand and the domestic price level at time t; r is the domestic nominal interest rate; and $e(t)$ is the logarithm of the exchange rate, that is, the price in home currency of one unit of foreign money. An increase in e means that domestic money depreciates. Here, y and $\tilde{p}$ are the logarithms of the domestic commodity supply and foreign price level; and $\tilde{r}$ is the foreign nominal rate of interest and m is the logarithm of the nominal stock of money. Equation (9.2.1) implies that home aggregate demand is a decreasing function of the home-to-foreign price ratio and of the expected real rate of interest. Equation (9.2.2) tells that excess demand for goods and services drives price inflation. Equation (9.2.3) is a standard Keynesian LM schedule that relates the demand for money to real money and the nominal interest rate. Equation (9.2.4) is an arbitrage condition in asset markets which says that the home interest rate will exceed the foreign interest rate by an amount exactly equal to the expected rate of depreciation in the home country. The model consists of four equations with four endogenous variables, describing the evolution of a small open economy with perfect foresight. Here, by perfect foresight it means that the rate of inflation individuals expect equals the actual rate

$$p(t+1) - p(t) = \log\left(\frac{P(t+1)}{P(t)}\right) \approx \frac{P(t+1)}{P(t)} - 1,$$

for small changes in the actual level of prices, $P(t)$. It should be noted that this model has different variations.[6]

To analyze the behavior of the model, we first eliminate two variables and reduce the dynamics to a 2-dimensional problem. We can eliminate $y^d(t)$ and $r(t)$ by substituting respectively (9.2.1) into (9.2.2) and (9.2.4) into (9.2.3). We thus have

$$\begin{aligned}&(1-\alpha\sigma)p(t+1)-(1-\alpha\sigma+\alpha\delta)p(t)\\&\quad=(\delta+\sigma)\alpha e(t)+\alpha\delta\tilde{p}-\alpha\sigma r^*-\alpha\sigma e(t+1),\\&\lambda e(t+1)-\lambda e(t)=p(t)+\phi y-\lambda\tilde{r}-m.\end{aligned}\tag{9.2.5}$$

The system has a unique equilibrium point

$$\begin{aligned}&e^*=p^*-\tilde{p}+\frac{y+\sigma\tilde{r}}{\delta},\\&p^*=\lambda\tilde{r}+m-\phi y\end{aligned}\tag{9.2.6}$$

At equilibrium, we also have $r(t)=\tilde{r}$ and $y^d(t)=\tilde{y}$. At equilibrium the domestic price level is an increasing function of the exogenous variables m and $\tilde{r}$, a decreasing function of y; the exchange rate is an increasing function of m and $\tilde{r}$, a decreasing function of $\tilde{p}$. We may transform the dynamical system (9.2.5) into a homogeneous one by introducing the following variables

$$x_1(t)\equiv e(t)-e^*,\quad x_2(t)\equiv p(t)-p^*.$$

Under the above transformation, we see that the dynamic system (9.2.5) can be written in the following form

$$\begin{bmatrix}x_1(t+1)\\x_2(t+1)\end{bmatrix}=\begin{bmatrix}1 & \dfrac{1}{\lambda}\\ \dfrac{\alpha\delta}{1-\alpha\sigma} & 1-\dfrac{\alpha\delta+\alpha\sigma/\lambda}{1-\alpha\sigma}\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}\tag{9.2.7}$$

or in matrix form, $x(t+1)=Ax(t)$. The characteristic equation is

$$z(\rho)=(\rho-\rho_1)(\rho-\rho_2)=\rho^2-(Traec\,A)\rho+Det\,A,$$

where

$$\begin{aligned}&Traec\,A=\rho_1+\rho_2=2-\frac{\alpha\delta+\alpha\sigma/\lambda}{1-\alpha\sigma},\\&Det\,A=\rho_1\rho_2=1-\frac{\alpha\delta}{1-\alpha\sigma}\left(\delta+\frac{\alpha}{\lambda}\right)-\frac{\alpha\delta}{\lambda(1-\alpha\delta)}.\end{aligned}$$

We calculate

$$\Delta = (TraecA)^2 - 4DetA = \left[\frac{\alpha}{1-\alpha\sigma}\left(\delta + \frac{\alpha}{\lambda}\right)\right]^2 + \frac{4\alpha\delta}{\lambda(1-\alpha\delta)},$$

$$z(1) = (1-\rho_1)(1-\rho_2) = -\frac{\alpha\delta}{\lambda(1-\alpha\delta)},$$

$$z(-1) = (1+\rho_1)(1+\rho_2) = 4 - \frac{\alpha}{1-\alpha\sigma}\left[2\left(\delta + \frac{\alpha}{\lambda}\right) + \frac{\alpha}{\lambda}\right].$$

If $1-\alpha\sigma > 0$, i.e. $\alpha < 1/\sigma$, the eigenvalues are real (because of $\Delta > 0$), lie on opposites of 1 (because $z(1) < 0$) and their sum is less than 2 (because of $TraceA < 2$). If the equilibrium point is a saddle, we need the eigenvalues to lie on different sides of -1, that is

$$z(-1) > 0 \Leftrightarrow \alpha > \frac{4}{\left(2+\lambda^{-1}\right)(\delta+2\sigma)}.$$

We conclude that if the coefficient α is small enough (price adjustment is sufficiently low), the steady state of the system is a saddle. In the remaining of this section, we require α to be sufficiently small so that the two eigenvalues satisfy: $\rho_1 > 1$ and $|\rho_2| < 1$.

The general solution may be written as

$$e(t) = e^* + c_1 h_{11}\rho_1^t + c_2 h_{21}\rho_2^t,$$
$$p(t) = p^* + c_1\rho_1^t + c_2\rho_2^t, \tag{9.2.8}$$

where $(h_{ii}, 1)^T$ is eigenvector associated with the eigenvalue ρ_i, for $i = 1, 2$. We are interested in a special path which is not explosive and a situation in which for the given level of price the exchange rate adjusts as needed to keep the system on the unique convergent path. To exclude explosive paths, we take $c_1 = 0$. Then c_2 can be determined by $c_2 = p(0) - p^*$. It is straightforward to show that (9.2.7) can be written as

$$e(t) = e^* + h_{21}(p(t) - p^*).$$

It is direct to check $h_{21} < 0$ by

$$Ah_2 = \rho_2 h_2, |\rho_2| < 1.$$

The equation thus describes the negatively sloped straight line through the equilibrium point $S = (e^*, p^*)$.

We may apply this equation to examine impact of changes in some parameters, say, m on the dynamics of the system. An increase in m will shift the equilibrium point. Equation (9.2.5) shows that the equilibrium values of the two variables will rise by an equal amount, from $S = (e^*, p^*)$ to $\hat{S} = (\hat{e}, \hat{p})$ in the long term.

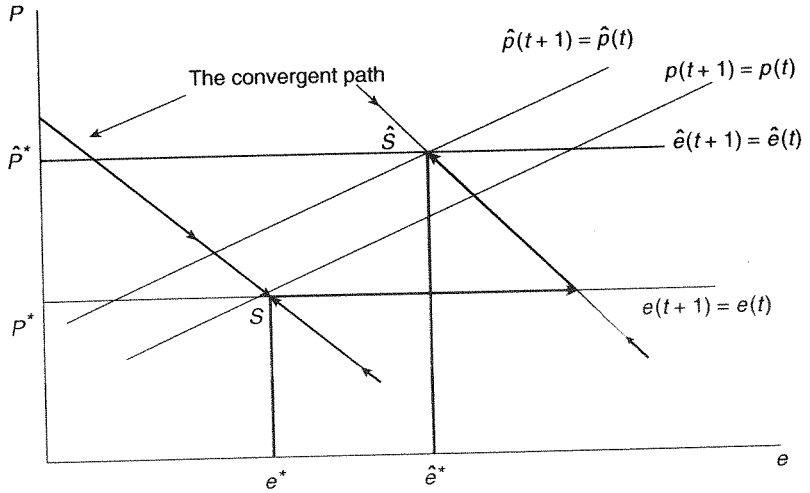


Figure 9.2.1 The overshooting adjustment process.

As price is assumed sticky, the adjustment process is depicted as in Figure 9.2.1. A monetary expansion results in an immediate depreciation of the currency and sustained the inflation as the price level gradually adjusts upward. In initial stage, the exchange rates overshoot its long-run equilibrium value, so that inflation is actually accompanied by a gradual depreciation of the currency during the transition. This happens as the presumed stickiness of output prices puts the full burden of the immediate adjustment on the exchange rate. The instantaneous depreciation produces disequilibrium in the goods market which adjusts over time as output prices adjusts. Following the sudden depreciation, and with output prices constant, domestic goods become cheaper relative to foreign goods. This leads to an excess demand for domestic output and to a gradual increase in domestic prices.

9.3 A small open-country economy with the CIA approach

The household behavior in the Dornbusch exchange-rate model in the previous section is not based on maximization. There are some trade models for a small country with a fixed commodity supply within an open-country model with perfect-foresight dynamic optimization.[7] Since the interest rate is fixed for a small country and the gap does not vary, the models predict monotonic movements of foreign asset holdings. The rate of time preference in the small economy equals the given world rate of interest is the standard assumption in the literature of a small open economy, based on the Ramsey approach.[8] If this assumption were not accepted, the domestic agent would end up either in infinite debt or infinite credit to the rest of the world.[9] This section is concerned with an economy similar to the small-country trade models.[10]

9.3.1 The model with the CIA approach

There is a single, perishable, consumption good, and households' financial wealth is divided between domestic fiat money and internationally traded bonds denominated in foreign currency. The economy is small, and it can influence neither the foreign currency price of the consumption good nor the world bond rate. The exchange rate is allowed to float freely by the monetary authority. As foreigners do not hold domestic money, the exchange rate adjusts to maintain equality between the real money supply and domestic real money demand. Perfect foresight is assumed. The foreign price of traded goods is given in the world market. The domestic residents may hold two assets, domestic money and a traded world bond.[11] We neglect transport cost, customs, or any other possible impediments to trade. We have perfect mobility of goods.

For each good, the law of one price holds. We have absolute purchasing power parity (PPP), which means that, measured in the same currency, the same basket of goods costs the same at home and abroad. We thus have

$$P(t) = E(t)\tilde{P}(t),$$

where $P(t)$ is the domestic price level, $\tilde{P}(t)(=1)$ is the foreign price level measured in foreign currency, and $E(t)$ is the exchange rate. The assumption of the PPP implies the following relation

$$\pi(t) = \tilde{\pi} + e(t), \tag{9.3.1}$$

where $\pi(t)$ and $\tilde{\pi}$ $(=0)$ are rates of inflation of the good, respectively, in domestic currency and foreign currency, and

$$e(t) \equiv \frac{\dot{E}(t)}{E(t)},$$

is rate of exchange depreciation of domestic currency. Assume that $\tilde{\pi}$ is fixed in the world market. Under free trade the rate of inflation in the domestic economy is equal to the world rate of inflation plus the rate of depreciation of domestic currency. There is a traded world bond with uncovered interest parity (UIP) holding at any point of time

$$i(t) = \tilde{i} + e(t), \tag{9.3.2}$$

where $i(t)$ and $\tilde{i}$ are respectively domestic and (fixed) foreign nominal interest rates.[12]

At each moment, the household allocates its disposable income between current expenditure and saving. Real output, f, is taken to be exogenous and fixed. We assume that the government's inflation tax, $e(t)m(t)$, is spent upon unproductive public services. The real disposable income is given by

$$\hat{y}(t) = f + a(t) + (i(t) - \pi(t))b(t) - \pi(t)m(t), \tag{9.3.3}$$

where

$$a \equiv m + b, \quad m \equiv \frac{M}{P}, \quad b \equiv \frac{EB}{P} = \frac{B}{\tilde{P}}.$$

The budget constraint is given by $c(t) + s(t) = \hat{y}(t)$. As in Section 5.2, the liquidity constraint of the household is formed as $m(t) = \chi c(t)$, where χ is a positive parameter.

Inserting $m(t) = \chi c(t)$ in the definition of $\hat{y}(t)$ and then substituting the resulted equation into $c(t) + s(t) = \hat{y}(t)$, we have

$$\hat{y} = f + a + (i - \pi)b - \chi \pi c. \tag{9.3.4}$$

We assume that at each point of time consumers' preferences over money, leisure time, consumption and saving can be represented by the following utility function: $U = c^{\xi_0} s^{\lambda_0}$. Maximizing $U(t)$ subject to the budget constraints yields

$$(1 + \chi\pi)c = \xi \bar{y}, \quad s = \lambda \bar{y}, \tag{9.3.5}$$

where

$$\bar{y} = f + a + (i - \pi)b, \quad \xi \equiv \rho \xi_0, \quad \lambda \equiv \rho \lambda_0, \quad \rho \equiv \frac{1}{\xi_0 + \lambda_0}.$$

The real wealth changes as follows

$$\dot{a}(t) = s(t) - a(t). \tag{9.3.6}$$

9.3.2 Dynamics and equilibrium

Substituting $i(t) - \pi(t) = i_0 \equiv \tilde{i} - \tilde{\pi}$ into the definition of $\bar{y}$ we have

$$\bar{y} = f + m + (1 + i_0)b. \tag{9.3.7}$$

Substituting $s = \lambda \bar{y}$ and equation (9.3.7) into equation (9.3.6), we have

$$\dot{b} = \lambda f + (\lambda i_0 - \xi)b - \xi m - \dot{m}. \tag{9.3.8}$$

Substituting $\pi = -\dot{m}/m$ into $(1 + \chi\pi)c = \xi \bar{y}$ yields

$$\dot{m} = \frac{(\chi - \xi)m - \xi f + (1 + i_0)\xi b}{\chi}, \tag{9.3.9}$$

where we use $m(t) = \chi c(t)$ and equation (9.3.7).

We conclude that the two linear differential equations (9.3.8) and (9.3.9) have two variables, $b(t)$ and $m(t)$. With proper initial conditions, we can thus determine

the two variables. Once we solve the two variables, the other variables are determined by the following procedure: $\pi = -\dot{m}/m \to \bar{y}$ by (9.3.7) $\to c$ and s by (9.3.5) $\to a = b + m$.

At steady state we have

$$\begin{aligned} &\lambda f + (\lambda i_0 - \xi) b - \xi m = 0, \\ &(\chi - \xi) m - \xi f + (1 + i_0)\xi b = 0. \end{aligned} \tag{9.3.10}$$

We solve equations (9.3.10) as

$$\begin{aligned} m &= \frac{\lambda f}{\xi} + \frac{(\lambda i_0 - \xi)\xi f}{(\chi - \xi)(\lambda f + \lambda i_0 - \xi) + (1 + i_0)\xi^2}, \\ b &= \frac{\xi^2 f}{(\chi - \xi)(\lambda f + \lambda i_0 - \xi) + (1 + i_0)\xi^2}. \end{aligned} \tag{9.3.11}$$

The two eigenvalues, $\phi_{1,2}$, are given by

$$\phi_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2},$$

where

$$\begin{aligned} a_1 &\equiv \frac{(2 + i_0)\xi}{\chi} - \lambda i_0 - \lambda, \\ a_0 &\equiv \frac{(1 + i_0)\xi^2 + (\xi - \lambda i_0)(\xi - \chi)}{\chi}. \end{aligned}$$

For convenience of discussion, we assume that $\chi = 1$ and i_0 is small. As $\xi = 1 - \lambda$, we have

$$a_1 = \xi + (\xi - \lambda)(1 + i_0), \quad a_0 = (\xi - \lambda)(\xi + i_0).$$

If $\xi > \lambda$, we have $a_1 > 0$ and $a_0 > 0$. The steady state is stable. If $\xi = \lambda$, we have $a_1 > 0$ and $a_0 = 0$. One eigenvalue is negative and the other one is equal to zero. If $\xi < \lambda$, we have $a_0 < 0$. The steady state is unstable.

9.3.3 Comparative statics analysis

As we explicitly solved the steady state, it is straightforward to analyze how change in a parameter affects the steady state. First, we examine effects of change in the foreign nominal interest rate, i_0. In this section, we require $\chi = 1$.

From equations (9.3.11), we have

$$\frac{dm}{di_0} = \frac{(\lambda^3 f + \xi^2)\xi f}{\left[(\lambda f + \lambda i_0 - \xi)\lambda + (1+i_0)\xi^2\right]^2} > 0,$$

$$\frac{db}{di_0} = -\frac{(\lambda^2 + \xi^2)\xi^2 f}{\left[(\lambda f + \lambda i_0 - \xi)\lambda + (1+i_0)\xi^2\right]^2} < 0.$$

As the foreign nominal interest rate rises, the real money balance rises and real bonds is reduced.[13] As $Pm = M$, we have

$$\frac{dP}{di_0} = -\frac{P}{m}\frac{dm}{di_0} < 0.$$

The price of goods fall. From the above equations, we have

$$\frac{dB}{di_0} = P\frac{db}{di_0} + b\frac{dP}{di_0} < 0.$$

From $a = m + b$, we have

$$\frac{da}{di_0} = \frac{\lambda^2 f + (\xi - \lambda)\xi}{\left[(\lambda f + \lambda i_0 - \xi)\lambda + (1+i_0)\xi^2\right]^2}\lambda\xi f.$$

We see that the sign of da/di_0 is the same as that of $\lambda^2 f + (\xi - \lambda)\xi$. From $m = \chi c$, the consumption level is increased.

9.4 A small open-country economy with the MIUF approach

The model of the previous section is defined with the CIA approach. This section examines the same issue but with the MIUF approach.

9.4.1 The model with the MIUF approach

Like in Section 9.3, the economy produces a single traded good with a single factor of production, labor. Perfect foresight is assumed. It has two assets, domestic money and traded bonds. The economy consists of consumers, firms and the government. The foreign price of traded goods is given in the world market. The domestic residents may hold two assets, domestic money and a traded world bond. We neglect transport cost, customs, or any other possible impediments to trade. We have perfect mobility of goods. For each good the law of one price holds. We have absolute purchasing power parity (PPP), which means that, measured in the same currency, the same basket of goods costs the same at home and abroad. Thus, $P(t) = E(t)\tilde{P}(t)$, where $P(t)$ is the domestic price level, $\tilde{P}(t)$ is the foreign price

level measured in foreign currency, and $E(t)$ is the exchange rate. The assumption of the PPP implies the following relation

$$\pi(t) = \tilde{\pi}(t) + e(t), \tag{9.4.1}$$

where $\pi(t)$ and $\tilde{\pi}(t)$ are rates of inflation of the good, respectively, in domestic currency and foreign currency, and $e(t) \equiv \dot{E}(t)/E(t)$ is rate of exchange depreciation of domestic currency. Here, $\tilde{\pi}(t)$ is fixed in the world market. Under free trade the rate of inflation in the domestic economy is equal to the world rate of inflation plus the rate of depreciation of domestic currency. There is a traded world bond with uncovered interest parity (UIP) holding at any point of time

$$i(t) = \tilde{i}(t) + e(t), \tag{9.4.2}$$

where $i(t)$ and $\tilde{i}(t)$ are respectively domestic and foreign nominal interest rates.

The household is endowed with 1 unit of time and choose endogenously to have leisure and work. We use a linear production function $f(t) = T(t)$, where $T(t)$ is the work time. Let $w(t)$ denote the real wage income. The zero profit of firms implies $w(t) = T(t)$. The consumer chooses his level of consumption, labor supply, real money balances, and holdings of the traded bond. His nominal disposable income is given as the sum of the current income and wealth in nominal terms, defined as follows.

$$\begin{aligned}\hat{y}_m(t) = {} & P(t)T(t) + a_m(t) + i(t)E(t)B(t) - P(t)\tau(t) - \pi(t)E(t)B(t) \\ & - \pi(t)M(t),\end{aligned} \tag{9.4.3}$$

where $B(t)$ is nominal stock of traded bonds, $\tau(t)$ is the real lump taxes, $a_m(t)$ is the wealth in domestic currency defined as

$$a_m(t) \equiv M(t) + E(t)B(t),$$

where $M(t)$ is nominal money balances. The real disposable income is given by

$$\hat{y}(t) = T(t) + a(t) + (i(t) - \pi(t))b(t) - \pi(t)m(t) - \tau(t), \tag{9.4.4}$$

where

$$\hat{y} \equiv \frac{\hat{y}_m}{P}, \quad a \equiv \frac{a_m}{P} = m + b, \quad m \equiv \frac{M}{P}, \quad b \equiv \frac{EB}{P} = \frac{B}{\tilde{P}}.$$

The budget constraint is given by

$$(1 + i(t) - \pi(t))m(t) + c(t) + s(t) = \hat{y}(t).$$

Inserting $T(t) = 1 - T_h(t)$ into the definition of $\hat{y}(t)$ and then substituting the resulted equation into the budget constraint, we have

$$im + c + s + T_h = y_a \equiv 1 + (1 + i - \pi)b - \tau. \tag{9.4.5}$$

We assume that at each point of time consumers' preferences over money, leisure time, consumption and saving can be represented by the following utility function

$$U(t) = m^{\varepsilon_0}(t) T_h^{\sigma_0}(t) c^{\xi_0}(t) s^{\lambda_0}(t), \quad \varepsilon_0, \sigma_0, \xi_0, \lambda_0 > 0.$$

The consumers' problem is to choose current consumption, leisure, and savings in such a way that utility levels are maximized. Maximizing $U(t)$ subject to the budget constraints yields

$$im = \varepsilon y_a, \quad T_h = \sigma y_a, \quad c = \xi y_a, \quad s = \lambda y_a, \tag{9.4.6}$$

where

$$\varepsilon \equiv \rho\varepsilon_0, \quad \sigma \equiv \rho\sigma_0, \quad \xi \equiv \rho\xi_0, \quad \lambda \equiv \rho\lambda_0, \quad \rho \equiv \frac{1}{\varepsilon_0 + \sigma_0 + \xi_0 + \lambda_0}.$$

The real wealth changes as follows[14]

$$\dot{a}(t) = s(t) - a(t). \tag{9.4.7}$$

The budget constraint of the government is given by

$$\dot{m}(t) + \dot{b}_g(t) = g(t) + \left(\tilde{i}(t) - \tilde{\pi}\right) b_g(t) - \pi(t) m(t) - \tau(t), \tag{9.4.8}$$

where $b_g(t)$ is stock of traded bonds issued by the domestic government and $g(t)$ is real government expenditure. From equations (9.4.7) and (9.4.8), we have

$$\dot{d}(t) = f(t) + \left(\tilde{i}(t) - \tilde{\pi}(t)\right) d(t) - c(t) - g(t) - \left(1 + \tilde{i}(t) - \tilde{\pi}(t)\right) m(t), \tag{9.4.9}$$

where $d(t) (\equiv b(t) - b_g(t))$ is the net stock of traded bonds and we use

$$im + c + s = f + (1 + i - \pi) b - \tau.$$

Equation (9.4.9) implies that the balance of payments on current account equals the balance of trade (i.e. $f - c - g$) minus the cost of holding money, $\left(1 + \tilde{i} - \tilde{\pi}\right) m$, plus the real interest earned on foreign bond holdings. If $d(t) > (<) 0$, the country is a creditor (debtor).

The government supplies the domestic nominal money at a fixed growth rate, μ, and continually balances its budget with lump-sum taxes, that is

$$\dot{m}(t) = (\mu - \pi(t)) m(t),$$
$$\tau(t) = \mu m(t) + g(t) + \left(\tilde{i}(t) - \tilde{\pi}(t)\right) b_g(t). \tag{9.4.10}$$

We assume that $g(t), \tilde{\pi}(t), \tilde{i}(t), \mu$ and $b_g(t)$ are exogenous.

9.4.2 The dynamics and steady state

Substituting $i(t)-\pi(t)=i_0(t)\equiv\tilde{i}(t)-\tilde{\pi}(t)$ into equation (9.4.5), we have

$$y_a=1+b+i_0 d-\mu m-g, \tag{9.4.11}$$

where we also use the second equation in (9.4.10). From $f=1-T_h$ and equations (9.4.6), we have $f=1-\sigma y_a$. Substitute this equation and $c(t)=\xi y_a(t)$ into equation (9.4.9), we have

$$\dot{d}=\overline{\sigma}-\sigma_0\left(d+b_g\right)-(1+i_0-\sigma_0\mu)m+\hat{\sigma}i_0 d-\overline{\sigma}g, \tag{9.4.12}$$

where we use equation (9.4.11) and

$$\hat{\sigma}\equiv 1-\sigma_0(=\lambda+\varepsilon)>0, \sigma_0\equiv\sigma+\xi>0.$$

Substituting $\pi(t)=\mu-\dot{m}(t)/m(t)$ into $(\pi(t)+i_0(t))m(t)=\varepsilon y_a(t)$ yields

$$\dot{m}=-\varepsilon-\varepsilon\left(d+b_g\right)-\varepsilon i_0 d+(i_0+\varepsilon\mu+\mu)m+\varepsilon g, \tag{9.4.13}$$

where we use equation (9.4.11).

We conclude that for any given $g(t),\tilde{\pi}(t),\tilde{i}(t),\mu$ and $b_g(t)$, the two linear differential equations (9.4.12) and (9.4.13) contain two variables, $d(t)$, and $m(t)$. With proper initial conditions, we can explicitly solve the linear equations. Once we solve the two variables, the other variables are determined by the following procedure: $a=m+b\rightarrow\pi=\mu-\dot{m}/m\rightarrow y_a$ by (9.4.11) $\rightarrow T_h$, c and s by (9.4.6) $\rightarrow T=1-T_h\rightarrow f=T\rightarrow b=d+b_g\rightarrow e=\pi-\tilde{\pi}\rightarrow E=E(0)\exp\left(\int_0^t e(t)dt\right)\rightarrow i=\tilde{i}+e\rightarrow P=E\tilde{P}\rightarrow a_m=Pa\rightarrow\hat{y}_m=P\hat{y}$.

According to this procedure, we can simulate the model with proper initial conditions and given functions, $g(t),\tilde{\pi}(t),\tilde{i}(t),\mu(t)$ and $b_g(t)$.

We now examine behavior of the model for the special case when $g(t),\tilde{\pi}(t)$, $\tilde{i}(t)$, μ and $b_g(t)$ are all constant. The dynamics are described by

$$\begin{aligned}\dot{d}&=\hat{\sigma}_d+(\hat{\sigma}i_0-\sigma_0)d-\mu_d m,\\ \dot{m}&=-\hat{\sigma}_m-(1+i_0)\varepsilon d+\mu_m m,\end{aligned} \tag{9.4.14}$$

where

$$\hat{\sigma}_d\equiv(1-g)\hat{\sigma}-\sigma_0 b_g, \hat{\sigma}_m\equiv\varepsilon\left(1-g+b_g\right)>0,$$

$$\mu_d\equiv 1+i_0-\sigma_0\mu, \mu_m\equiv i_0+\varepsilon\mu+\mu>0.$$

We have $\overline{\sigma}_m>0$ as we require $1-g>0$.[15] The unique steady state is given by

$$d=\frac{\mu_d\hat{\sigma}_m+\mu_m\hat{\sigma}_d}{(\hat{\sigma}i_0-\sigma_0)\mu_m-(1+i_0)\mu_d\varepsilon}, m=\frac{(\hat{\sigma}i_0-\sigma_0)\hat{\sigma}_m+\varepsilon\mu_m\hat{\sigma}_d(1+i_0)}{(\hat{\sigma}i_0-\sigma_0)\mu_m-(1+i_0)\mu_d\varepsilon}. \tag{9.4.15}$$

It is straightforward to confirm the following relation:

$$(\hat{\sigma} i_0 - \sigma_0)\mu_m - (1+i_0)\mu_d \varepsilon = (\lambda - \sigma_0) i_0^2 - 2 i_0 \sigma_0 - \sigma_0 \mu - i_0 \sigma_0 \mu - \varepsilon - 2\varepsilon i_0.$$

From the definitions of i_0 and σ_0, it is reasonable to require $(\lambda - \sigma_0) i_0^2$ to be small. Under this requirement, the denominators of the right-hand sides of the two equations are negative. From

$$(\hat{\sigma} i_0 - \sigma_0)\hat{\sigma}_m + \varepsilon \mu_m \hat{\sigma}_d (1+i_0)$$
$$= \left[(i_0 + \mu_m + i_0 \mu_m)\hat{\sigma} - \sigma_0 - \frac{(1+i_0)\mu_m b_g}{1-g+b_g} \right] \frac{\varepsilon}{1-g+b_g},$$

we see that if $(i_0 + \mu_m + i_0 \mu_m)\hat{\sigma} \leq \sigma_0$ (which is guaranteed if i_0 and μ are small), then the number is negative. We thus conclude that m is positive if i_0 and μ are small. If $\mu_d \hat{\sigma}_m + \mu_m \hat{\sigma}_d < (>) 0$, then d is positive (negative).

The two eigenvalues, $\phi_{1,2}$, are given by

$$\phi_{1,2} = \frac{a_1 \pm \sqrt{a_1^2 - 4a_0}}{2},$$

where

$$a_1 \equiv \hat{\sigma} i_0 - \sigma_0 + \mu_m,$$
$$a_0 \equiv (\hat{\sigma} i_0 - \sigma_0)\mu_m - (1+i_0)\varepsilon \mu_d.$$

If i_0 and μ are very small, then we have $a_0 < 0$. The steady state is unstable.

9.4.3 Comparative static analysis

The inflation policy

First, we examine effects of a change in the inflation policy. Taking derivatives of equations (9.4.15) with respect to μ yields

$$\frac{1}{d}\frac{dd}{d\mu} = \frac{-\sigma_0 \hat{\sigma}_m + (1+\varepsilon)\hat{\sigma}_d}{\mu_d \hat{\sigma}_m + \mu_m \hat{\sigma}_d} - \frac{(\hat{\sigma} i_0 - \sigma_0)(1+\varepsilon) + (1+i_0)\sigma_0 \varepsilon}{(\hat{\sigma} i_0 - \sigma_0)\mu_m - (1+i_0)\mu_d \varepsilon},$$
$$\frac{1}{m}\frac{dm}{d\mu} = \frac{\varepsilon \hat{\sigma}_d (1+i_0)(1+\varepsilon)}{(\hat{\sigma} i_0 - \sigma_0)\mu_m - (1+i_0)\mu_d \varepsilon} - \frac{(\hat{\sigma} i_0 - \sigma_0)(1+\varepsilon) + (1+i_0)\sigma_0 \varepsilon}{(\hat{\sigma} i_0 - \sigma_0)\mu_m - (1+i_0)\mu_d \varepsilon}.$$

If i_0 and μ are small, then we have $dm/d\mu < 0$. As the inflation rate rises, the net stock of traded bonds is ambiguous and the real money balance falls. From $b = d + b_g$ and $a = m + b$, we calculate

$$a = \frac{(\hat{\sigma} i_0 - \sigma_0 + \mu_d)\hat{\sigma}_m + \mu_m \hat{\sigma}_d (1 + \varepsilon + \varepsilon i_0)}{(\hat{\sigma} i_0 - \sigma_0)\mu_m - (1+i_0)\mu_d \varepsilon} + b_g.$$

From this equation and $y_h = a/\lambda$ we have

$$\frac{1}{\lambda}\frac{dy_a}{d\mu} = \frac{da}{d\mu} = (a - b_g)$$
$$\left[\frac{-\sigma_0\hat{\sigma}_m + (1+\varepsilon)(1+\varepsilon+\varepsilon i_0)\hat{\sigma}_d}{(\hat{\sigma} i_0 - \sigma_0 + \mu_d)\hat{\sigma}_m + \mu_m\hat{\sigma}_d(1+\varepsilon+\varepsilon i_0)} - \frac{(\hat{\sigma} i_0 - \sigma_0)(1+\varepsilon) + (1+i_0)\sigma_0\varepsilon}{(\hat{\sigma} i_0 - \sigma_0)\mu_m - (1+i_0)\mu_d\varepsilon}\right].$$

The effects are ambiguous. From $T_h = \sigma y_a$ and $c = \xi y_a$, we get the impact on the leisure and the consumption level.

9.4.4 The government's real expenditure

We now study the government's real expenditure. Taking derivatives of equations (9.4.15) with respect to g yields

$$\frac{dd}{dg} = \frac{-\varepsilon\mu_d - \mu_m\hat{\sigma}}{(\hat{\sigma} i_0 - \sigma_0)\mu_m - (1+i_0)\mu_d\varepsilon},$$
$$\frac{dm}{dg} = -\frac{(\hat{\sigma} i_0 - \sigma_0) + \mu_m\hat{\sigma}(1+i_0)}{(\hat{\sigma} i_0 - \sigma_0)\mu_m - (1+i_0)\mu_d\varepsilon}\varepsilon.$$

We will not further examine how the equilibrium is affected by the parameters as expressions are tedious.

9.5 A multi-country economic model

This section develops a multi-country model with money, based on the multi-country model proposed by Ikeda and Ono (1992). The main difference is that this section uses the alternative approach to households used in this study, while Ikeda and Ono use the Ramsey approach.

9.5.1 The global economy

The world consists of J countries, indexed by $j = 1, \ldots, J$. Country j's population, N_j, is constant. The world produces a single tractable commodity. For illustration, we neglect capital and assume a constant commodity supply in each country. The representative household receives constant commodity supply, f_j, per unit time. Let $b_j(t)$ denote household j's foreign asset holdings (in real terms) and $r(t)$ the world rate of interest on $b_j(t)$. We assume that each country's money is a nontradeable asset.

The consumer chooses his level of consumption, real money balances, holdings of the traded bond. Household j's real disposable income is given by

$$\hat{y}(t) = f_j + a_j(t) + r(t)b_j(t) - \pi_j(t)m_j(t) + \tau_j(t), \tag{9.5.1}$$

where $\tau_j(t)$ is country j's government transfer and

$$a_j(t) \equiv m_j(t) + b_j(t), \quad m_j(t) \equiv \frac{M_j(t)}{P_j(t)},$$

where $P_j(t)$ is the price level in country j and $M_j(t)$ is the nominal money stock per household. The representative household takes $\tau_j(t)$ as given. The variable, $\tau_j(t)$, satisfies the government budget constraint

$$\tau_j(t) = \frac{\mu_j M_j(t)}{P_j(t)} = \mu_j m_j(t), \tag{9.5.2}$$

where μ_j is the constant rate of monetary expansion. We also have

$$\dot{m}_j(t) = \big(\mu_j - \pi_j(t)\big) m_j(t). \tag{9.5.3}$$

It should be noted that since the commodity is homogeneous, exchange rate, $E_{ij}(t)$, (the value of currency j in terms of currency i) is represented by $P_i(t)/P_j(t)$ for $i,j = 1,\ldots,J$. Changes in exchange rates are given by

$$\frac{\dot{E}_{ij}(t)}{E_{ij}(t)} = \pi_i(t) - \pi_j(t). \tag{9.5.4}$$

The representative household's budget constraint is given by

$$(1 + r(t)) m_j(t) + c_j(t) + s_j(t) = \hat{y}_j(t).$$

Substituting equation (9.5.1) into the budget constraint yields

$$\big(r + \pi_j\big) m_j + c_j + s_j = \bar{y}_j \equiv f_j + (1+r) b_j + \tau_j, \tag{9.5.5}$$

where we also use $a_j = m_j + b_j$.

We assume that at each point of time consumers' preferences over leisure time, consumption and saving can be represented by the following utility function[16]

$$U_j(t) = m_j^{\varepsilon_{0j}}(t) c_j^{\xi_{0j}}(t) s_j^{\lambda_{0j}}(t), \quad \varepsilon_{0j}, \xi_{0j}, \lambda_{0j} > 0,$$

where ε_{0j} is household j's propensity to hold money, ξ_{0j} propensity to consume, and λ_{0j} propensity to own wealth. A representative consumer's problem is to choose real balance, current consumption, and savings in such a way that utility levels are maximized. Maximizing $U_j(t)$ subject to the budget constraint yields

$$\big(r + \pi_j\big) m_j = \varepsilon_j \bar{y}_j, c_j = \xi_j \bar{y}_j, s_j = \lambda_j \bar{y}_j, \tag{9.5.6}$$

where

$$\varepsilon_j \equiv \rho_j \varepsilon_{0j}, \xi_j \equiv \rho_j \xi_{0j}, \lambda_j \equiv \rho_j \lambda_{0j}, \rho_j \equiv \frac{1}{\varepsilon_{0_j} + \xi_{0j} + \lambda_{0j}}.$$

Household j's real wealth changes as follows[17]

$$\dot{a}_j(t) = s_j(t) - a_j(t). \tag{9.5.7}$$

The equilibrium conditions in the commodity and bond markets are respectively given by

$$\sum_{j=1}^{J} c_j(t) N_j = y \equiv \sum_{j=1}^{J} f_j N_j, \tag{9.5.8}$$

$$\sum_{j=1}^{J} b_j(t) N_j = 0. \tag{9.5.9}$$

9.5.2 The motion of the global economy and the steady state

Substituting $b_j = a_j - m_j$ and equations (9.5.2) into the definition of $\bar{y}_j(t)$ yields

$$\bar{y}_j = f_j + (1+r)(a_j - m_j) + \mu_j m_j. \tag{9.5.10}$$

Substituting $c_j = \xi_j \bar{y}_j$ into equation (9.5.8), we have

$$\sum_{j=1}^{J} \xi_j N_j \bar{y}_j = y.$$

Substituting equations (9.5.10) into the above equation, we solve

$$r = \Lambda_r(a, m) \equiv \frac{y_0 - \sum_{j=1}^{J} \mu_j \xi_j N_j m_j}{\sum_{j=1}^{J} (a_j - m_j) \xi_j N_j} - 1, \tag{9.5.11}$$

where

$$y_0 \equiv y - \sum_{j=1}^{J} \xi_j N_j f_j = \sum_{j=1}^{J} (1 - \xi_j) N_j f_j > 0.$$

The rate of interest, $r(t)$, can be expressed as functions of $a(t)$ and $m(t)$, where $a(t)$ and $m(t)$ are respectively the global distributions of national real wealth, $a_j(t)$,

and real money, $m_j(t)$. From equations (9.5.10) and (9.5.11), we can express $\bar{y}_j(t)$ as functions of $a(t)$ and $m(t)$ as follows

$$\bar{y}_j = f_j + (1+\Lambda_r)(a_j - m_j) + \mu_j m_j. \tag{9.5.12}$$

Substituting $s_j(t) = \lambda_j \bar{y}_j(t)$ and equations (9.5.12) into equations (9.5.7), we obtain

$$\dot{a}_j = \Lambda_{aj}(a,m) \equiv \lambda_j f_j + (\lambda_j + \lambda_j \Lambda_r - 1) a_j - (1 - \mu_j + \Lambda_r) \lambda_j m_j. \tag{9.5.13}$$

Substituting equation (9.5.3) into $(\Lambda_r + \pi_j) m_j = \varepsilon_j \bar{y}_j$, we obtain

$$\dot{m}_j = (\bar{\mu}_j \Lambda_r + \varepsilon_j \Lambda_r) m_j - \varepsilon_j f_j - (1+\Lambda_r)\varepsilon_j a_j, \tag{9.5.14}$$

in which we use equations (9.5.12) and $\bar{\mu}_j \equiv \mu_j - \varepsilon_j \mu_j + \varepsilon_j > 0$.

The dynamics consist of $2J$ differential equations, (9.5.13) and (9.5.14) with $2J$ variables, $a_j(t)$ and $m_j(t)$. The other variables can be represented as functions of a_j and m_j by the following procedure: r by (9.5.11) $\rightarrow \bar{y}$ by (9.5.10) $\rightarrow \pi_j, c_j$ and s_j by (9.5.6) $\rightarrow b_j = a_j - m_j$.

At steady state we have

$$\begin{aligned} &f_j + (\Lambda_r - \bar{\xi}_j) a_j - (\Lambda_r - \mu_j) m_j = 0, \\ &-f_j - (1+\Lambda_r) a_j + (\bar{\varepsilon}_j + \Lambda_r) m_j = 0, \end{aligned} \tag{9.5.15}$$

where

$$\bar{\xi}_j \equiv \frac{\xi_j}{\lambda_j}, \bar{\varepsilon}_j \equiv \frac{\mu_j}{\varepsilon_j} - \mu_j.$$

We solve a_j and m_j as functions of Λ_r as follows

$$\begin{aligned} a_j &= \frac{\theta_{aj}}{\Lambda_r + \theta_j}, \\ m_j &= \frac{\theta_{mj}}{\Lambda_r + \theta_j}, \end{aligned} \tag{9.5.16}$$

where

$$\theta_{aj} \equiv \frac{f_j \mu_j / \varepsilon_j}{1 - \mu_j/\varepsilon_j + \bar{\xi}_j}, \theta_{mj} \equiv \frac{(1+\bar{\xi}_j) f_j}{1 - \mu_j/\varepsilon_j + \bar{\xi}_j}, \theta_j \equiv \frac{\bar{\xi}_j \bar{\varepsilon}_j - \mu_j}{1 - \mu_j/\varepsilon_j + \bar{\xi}_j}.$$

Substituting equations (9.5.16) into equation (9.5.11), we have

$$\sum_{j=1}^{J} [\theta_{0j} + (\theta_{aj} - \theta_{mj}) \Lambda_r] \frac{\xi_j N_j}{\Lambda_r + \theta_j} = y_0, \tag{9.5.17}$$

where $\theta_{0j} \equiv \theta_{aj} + (\mu_j - 1)\theta_{mj}$. This equation contains a single variable, Λ_r. This is a nonlinear equation. Once we solve this equation, we can determine the steady

state of the global economy. We will not further examine the model because the problem is nonlinear and high dimensional and it is difficult to get new insights. The interested reader can examine behavior of the model by simulation as we have already provided the procedure to determine the motion of all the variables.

9.6 Money, capital, knowledge, trade, and growth

This chapter proposed a few models of trade. There is a vast literature on monetary growth and international trade. It can be seen that the OSG framework provides a useful framework for analyzing different issues related to growth, trade, and money. It is possible to introduce the ideas about multiple sectors, heterogeneous groups, multiple regions in each country, research, education, and human capital proposed in the previous chapters into the trade models in this chapter.[18] This chapter does not study capital accumulation and knowledge growth. A host of trade problems such as trade patterns, gains from trade, commercial policy, transaction corporations, direct foreign investment have been examined within economic systems with changeable returns.

APPENDIX

A.9.1 The IS-LM model for an open economy

The IS-LM model for an open economy is the same as the closed-economy model described in Section 5.2, except that the IS curve is affected by trade conditions. For an open economy, the equations for desired consumption and investment are the same as before

$$C = \xi_0 + \xi_Y(Y - \tau_0 - \tau_w Y) - \xi_r r, \tag{A.9.1.1}$$

$$I = i_0 - i_r r. \tag{A.9.1.2}$$

It is assumed that net exports, NX, are determined as follows

$$NX = \eta - \eta_Y Y + \tilde{\eta}_Y \tilde{Y} - \eta_r r + \tilde{\eta}_r \tilde{r}, \tag{A.9.1.3}$$

where $\eta, \eta_Y, \tilde{\eta}_Y, \eta_r$, and $\tilde{\eta}_r$ are positive numbers, and $\tilde{Y}$ and $\tilde{r}$ are foreign income and foreign rate of interest. Equation (A.9.1.3) says that the net exports are positively related to the foreign income and real rate of interest and negatively to the domestic income and real rate of interest. The goods market equilibrium condition now becomes

$$Y = C + I + G + NX. \tag{A.9.1.4}$$

This condition can also be written as $S = I + NX$, where S is desired savings. The IS curve is now given as

$$r = \alpha_{IS} - \beta_{IS} Y, \tag{A.9.1.5}$$

in which

$$\alpha_{IS} \equiv \frac{\xi_0 + i_0 + G - \xi_Y \tau_0 + \eta + \tilde{\eta}_Y \tilde{Y} + \tilde{\eta}_r \tilde{r}}{\xi_r + i_r + \eta_r}, \quad \beta_{IS} \equiv \frac{1 - (1 - \tau)\xi_Y + \eta_Y}{\xi_r + i_r + \eta_r} > 0.$$

For the open economy, the LM curve is the same as in the closed economy, that is, equation (5.2.10). General equilibrium in the open-economy IS-LM model is also found by first finding the intersection of the two curves. We can also similarly conduct the classical and Keynesian AD-AS analysis for the open economy as for the closed economy.

10 Money and economic complexity

> Technical skill is mastery of complexity while creativity is mastery of simplicity.[1]
>
> Chris Zeeman

To simplify complicated issues related to growth with money, this book proposes an alternative approach to the traditional approaches. The numerous models proposed in this study demonstrates that the approach in modeling household behavior proposed in study is effective in analyzing many dynamic issues, in comparison to, for instance, the Ramsey approach. When speaking about developing a new theory, I am aware of the following criteria of a theory proposed by Kuhn.[2]

> A theory should be: (1) accurate within domain; (2) consistent internally and with other currently accepted theories; (3) a broad scope allowing its consequences to extend beyond the particular observations, laws or subtheories it was initially designed to explain; (4) simple but able to bring order to the phenomena that in its absence would be individually isolated and, as a set, confused; and (5) fruitful to disclose new phenomena or previously unnoted relationships among those already known.[3]

These five characteristics – accuracy, consistency, scope, simplicity, and fruitfulness – for evaluating the adequacy of economic theory are important as they may help to make us aware of limitations of different economic theories.

This book presents a monetary growth theory based on the traditional monetary growth models. The main deviation from the traditional approaches is the alternative utility function, which helps us to solve many important problems with a consistent manner.[4] This book does not examine empirical aspects of monetary theory. There is a vast body of empirical research in monetary economics.[5] Different empirical studies give different answers to almost any important issue in monetary growth economics. It is simply beyond my ability to properly judge the validity of these empirical studies. We now mention a few straightforward directions for extending this book.

Functional generalizations

Given the rapid development of pure and applied mathematics, it is possible to make any economic theory in more general than its current form by generalizing some of its functional forms or "connections." For instance, utility and functions used in our analyses can take forms that are more general. One can directly apply advanced mathematics to generalize many results in this book.

Synthesizing the basic models in this book

This book proposed some monetary growth models. Like blood in the human body, money affects and is affected by almost every aspect of the economic system. Forms of its existence and function can exhibit a great variety, depending on, for instance, technology, economic developmental stages, institutional structures, and man's attitude towards the future. Money does not only provide services of the present but also plays the role of a connector of the present and the future. This book introduces the well-accepted formations of money in monetary growth theory to the analytical framework proposed by the author. In principle, almost all the models can be connected with each other in the sense that we can build a more general model which includes the models as special cases. For instance, Section 3.6 builds a monetary growth model with education through the MIUF approach. Whereas, Section 4.5 proposes a monetary growth model through research with the CIA. It is possible to construct a growth model with both education and research for knowledge growth. Similarly, we can build a general model both with urban economic structures and interregional interactions, by combining the urban model in Chapter 7 and the interregional model in Chapter 8. It can be seen that numerous various combinations are possible. Nevertheless, more general models are analytically complicated and few new insights can be obtained.

As Hahn (2000: 65) points out:

> ... any study of liquidity soon leads to the conclusion that if it is an important element in an economy's analysis, then many of theories of the economy are in need of change. For instance, the recognition of the importance of transaction costs leads us to conclude that agents will maximize, subject to a number of, rather than a single, budget constraints.

This book does not provide a definite answer to the question about the role of money in economic growth. As Dornbusch and Frenkel (1973) observed, "conflicting or ambiguous results derived from alternative theories are primarily the reflection of different hypotheses about functions of money." We tried to examine money from different perspectives.

Risks, uncertainty, expectations, and stochastic processes

This book builds models with certainty. In reality, decisions involve risks and uncertainties.

For instance, an important issue in monetary economics is related to announcement effects of anticipated monetary policy upon behavior of economic systems.[6] Basing on the literature on the topic,[7] we can examine the issues within the models proposed in this book. As the literature in monetary economics with risks, uncertainties and expectations is vast, it takes a long space even to make a basic introduction to the literature. We now introduce two well-known examples in monetary economics to show how expectations can be introduced into the monetary growth theory.

First, we are concerned with an empirical model of money and inflation due to Cagan (1956).[8] In his classical study, Cagan studied hyperinflations.[9] The Cagan model of hyperinflation is described by the following pair of equations

$$m(t) - p(t) = -\alpha\pi(t), \quad \alpha > 0,$$
$$\dot{\pi}^e(t) = \gamma\,(\dot{p}(t) - \pi^e(t)), \quad \gamma > 0,$$

where $m(t)$ is natural log of the nominal stock of money, $M(t)$, p is natural log of price level, $P(t)$, $\pi^e(t)$ is expected rate of inflation, and α and λ are parameters. In this model, output and real interest rate are fixed and wealth effects are ignored. The first equation describes continuous money market equilibrium, and the second equation expresses the usual form of the adaptive hypothesis about the expected rate of inflation. The coefficient

$$\alpha = \frac{1}{M}\frac{dM}{d\pi}$$

is interpreted as a semi-elasticity of demand. It measures the percentage change in the demand for money per percentage point change in the expected inflation rate. Supposing the nominal money stock remains constant, from the first equation we obtain $\dot{p} = \alpha\dot{\pi}^e$. Substituting the first equation in the Cagan model and $\dot{p} = \alpha\dot{\pi}^e$ into the second equation yields

$$\dot{p} = \frac{\gamma}{1-\alpha\gamma}(m-p).$$

We conclude that the adjustment of prices is stable if and only if $\alpha\gamma < 1$. This relation emphasizes the trade-off between the semi-elasticity of the demand for money with respect to inflationary expectations and the rate of adaptation of inflationary expectations. A highly sensitive demand for money function is compatible with stability only if inflationary expectations adapt sufficiently slowly to past inflation rates.

Another well-known model is the so-called Lucas-Sargent-Wallace (LSW) model.[10] The model and its variations are important for deriving the implications of the neoclassical and modern classical models and testing them. The model deals with interactions between the markets for commodities and money under the assumption that the markets are in equilibrium. The supply of output is represented

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by the expectational error-based Lucas supply rule as

$$D_t^s = \alpha Dy_{t-1} + \beta(p_t - p_t^e) + \mu_t, \quad \alpha, \beta > 0, \tag{10.1}$$

where all lower-case variables are in logs, the superscripts s and d represent respectively for supply and demand, y is output, Dy_t is the deviation of output in period t from its full employment level, y^f (that is, $Dy_t = y_t - y_t^f$), p is the price level, p^e is expected price level with expectations formed one period earlier, and μ is random term. The random have a zero expected value and are independent of the other variables in the model. The demand for output is specified as

$$y_t^d = \theta(m_t - p_t) + \eta_t, \quad \theta > 0, \tag{10.2}$$

where y_t^d is the aggregate demand, m is the nominal money supply, and η is a random term. The money supply function is specified as

$$m_t = m_0 + \gamma Dy_{t-1} + \xi_t, \quad \gamma < 0. \tag{10.3}$$

The equation implies that the monetary authority increases the money supply if output in the last period was below the full employment level. The equilibrium condition for the commodity market is

$$y_t = y_t^d = y_t^s. \tag{10.4}$$

This equation is equivalently represented by $Dy_t = Dy_t^d = Dy_t^s$. The assumption of rational expectations hypothesis is expressed as

$$p_t^e = Ep_t, \tag{10.5}$$

where Ep_t is the expected price conditional on information available in $t-1$. We have thus defined the LSW model which consists of (10.1)–(10.5). The basic question of this model is how the authority can manipulate the monetary policy, characterized by m_0 and γ, so that the output is maximized.[11]

Outside and inside money

This book is mainly concerned with models in which all money is assumed to be of the outside form, standing for non-interest-bearing government debt. In reality a large part of money supply is of the inside form, for instance, bank deposits – representing claims to the private sector. Aggregated over the private sector, outside money is part of net wealth of the private sector, whereas inside money is not. The importance of the distinction between inside and outside money was recognized early in the monetary growth literature.[12] As pointed out in Saving (1970: 83):

> Since the introduction of the distinction between inside and outside money it has been generally conceded that the classical argument concerning the

> balance effect holds only for outside money. ... Even in a system containing both inside and outside money, part of the classical argument breaks down in that money is no longer neutral.

In the last few decades, the technological changes have led to a widespread use of different forms of electronic money. It is expected that dynamics of inside and outside money should have been changed due to the economic development and technological changes. The models in this book neglect these dynamics. We may take account of these dynamics, for instance, according to Marimon *et al.* (2003) and Powers (2005).

Monetary policy rules

We did not make any comprehensive study of money supply. The assumption that the money supply is exogenous has been commonly accepted in the literature of theoretical monetary economics. Bain and Howells (2003: xi) observe:

> ... the dominant assumption almost everywhere that the money supply is exogenous, when clearly it is not – and everyone knows that it is not, although not everyone seems yet prepared to admit it, much less to admit the consequences that follow.

As pointed out by Walsh (2003: 4), although a large branch of the literature in monetary economics are concerned with the determinants of the demand for money, the focus in the literature has increasingly shifted towards the determinants of the supply of money.

> This change has occurred because, to a large extent, central banks operate today by employing a short-term interest rate as their policy operating target, with a deemphasis on the quantity of money. Such a focus reduces the importance of money-demand and money-supply analysis.[13]

In Section 4.4, we introduced Taylor's interest rate feedback rule into the CIA-OSG growth model. There are quite a few monetary policy rules suggested in the literature of monetary economics.[14] A monetary policy rule describes how the policy maker decides the money supply. Once a money supply function is specified, each period's supply is determined by the variables and parameters in the function. An example of a money supply rule is

$$m_t = m_0 + \gamma y_{t-1}, \quad \gamma < 0,$$

where all lower-case variables are in logs, y is output, m is the nominal money supply, and γ is a parameter. When $\gamma = 0$, the rule is suggested by Milton Friedman for an economy without growth.[15]

Another important direction in extending our approach is to examine how monetary policies are made in a global economy. For instance, Devereux *et al.* (2006)

examine an optimal monetary policy in a world with a dollar standard.[16] The model is based on an expected utility of agents

$$E\left(\frac{C^{1-\rho}}{1-\rho}+\chi\frac{M}{P}-\eta T\right)$$

in which C, M, P, and T are respectively an aggregate of home and foreign composite goods, M/P is real money balances, and labor supply, and ρ, χ, and η are the parameters. The paper shows that the US is essentially indifferent to exchange rate volatility, while the rest of the world places a high weight on exchange rate volatility. They apply concepts such as Nash equilibrium in game theory to examine how monetary policy game is played among nations.

Changes in transactions technology and other variables

Payment systems have gone through a major transformation in the last two decades in many parts of the world. The development of the technology to transfer information has increased substitutability between deposits and currency in transactions. It has been observed that the cost of transactions has been substantially reduced due to the technological changes. Innovations in financial markets have created and will continue to create new methods of transacting, such as credit cards, debit cards, instant cash machines, etc.[17] Institutional structures will affect how money interacts with the rest of the economy. Monetary institutions have been introduced into growth theory in different ways. We examined interactions among unemployment, growth and inflation in Chapter 5. It is important to introduce money into economic growth with disequilibrium.[18] Tax structures also impact upon effects of money on economic growth.[19]

We mentioned a few possible ways for extending our study. The list can be easily refined and continued. The number of questions raised is much more than the number of questions fully answered. More challenges are left for research in the future.

Notes

1 Money and growth theory

1 See Chapter 1 in Handa (2005) for various definitions of money.
2 Assets that actually perform the role of medium of payments vary over time.
3 Reprinted in Hume (1970).
4 Hume (1752). In the 1970 edition, pages 38–9.
5 After Pigou's seminal contribution, Patinkin (1956), Friedman (1956, 1969), Johnson (1969), and many others also made important contributions to the literature. See also Lucas (1996).
6 Fiat money is a type of money that has no intrinsic value, because it yields neither direct utility to consumers, nor it is an input to any production process.
7 See Bewley (1983).
8 See Walsh (2003: 2.1).
9 This book does not consider nonmonetary economic growth as Zhang (2005a, 2006a) comprehensively studied nonmonetary economic growth with capital, knowledge, leisure time, heterogeneous households and economic structure over time and space.
10 Real variables include, for instance, real output, human capital, knowledge, employment, real rates of interest, wages, and real exchange rates. Nominal variables include the inflation rate, nominal interest rates, nominal exchange rates, and the supply of money.
11 Sections 1.2–1.9 are based on Zhang (2006a: Chapters 1 and 2).
12 The description of behavior of producers and production sectors follows the traditional approach (e.g. Burmeister and Dobell, 1970; Azariadis, 1993; Zhang, 1999).
13 A production function $F(K,N)$ is called neoclassical if it satisfies the following conditions: (1) $F(K,N)$ is non-negative if K and N are non-negative; (2) $F(0,0)=0$; (3) marginal products, F_K and F_N are non-negative; (4) there exist second partial derivatives of F with respect to K and N; (5) the function is homogeneous of degree one: $F(\lambda K,\lambda N)=\lambda F(K,N)$, for all non-negative λ; (6) the function is strictly quasi-concave.
14 For the production function $F(K,N)$ we define the homogeneity of degree n for capital and labor inputs as follows: $F(\lambda K,\lambda N)=\lambda^n F(K,N)$, where λ is an arbitrary non-negative number. When $n=1$, we say that the production function has constant returns to scale. It is linearly homogeneous or homogeneous of degree one.
15 See Chiang (1984), Mas-Colell *et al.* (1995), and Simon and Blume (1994).
16 See Zhang (2005a).
17 The proof is referred to, for instance, Burmeister and Dobell (1970), or Zhang (2005a).
18 See Chapter 2 in Zhang (2005a) for the cases that the production function is taken on the CES form and the Leontief form.
19 It is possible to treat the parameters in the utility function as endogenous variables so that the labor supply curve is S-shaped and the system has multiple equilibria. See, for example, Dessing (2002, 2004) and Wetzell (2002) for further examinations. Combing the model in this section and the model with endogenous leisure in Section 4.2, we

can examine relations between government spending and capital accumulation with endogenous time preference – the topic is addressed by Chang *et al.* (1998a, 1998b) with the Ramsey approach.

20 Our following discussion on the relationship between the preference structure and utility function is referred to Barten and Böhm (1982).

21 Debreu (1959) and Rader (1963).

22 For instance, for smoothing out income fluctuations, providing for retirement and other expected future events.

23 Keynes (1936: 96).

24 This section is referred to Froyen (1999: 282–6).

25 Modigliani (1966).

26 Zhang (2005a) also discussed the permanent income hypothesis developed by Milton Friedman and possible relations between the permanent income hypothesis and the OSG approach.

27 Solow (2000: x).

28 Solow (1956) and Swan (1956). The Solow model is often called the Solow-Swan model because Swan's work is similar to Solow's seminal paper.

29 The description of behavior of producers and production sectors follows the traditional approach (e.g. Burmeister and Dobell (1970), Azariadis (1993) and Zhang (1999)).

30 *Macroeconomic Dynamics* (2006, Vol. 10). The early literature on the Ramsey model is referred to by Ramsey (1928), Cass (1965) and Koopmans (1965).

31 Although the acceptance of the Ramsey framework has not been based on any profound reason or empirical test, any alternative to such a well-accepted doctrine over a long time will rationally experience some "professional" rejections.

32 We will not provide a complete analysis of the model. Refer to Takayama (1985) and Romer (1996) in detail.

33 Samuelson (1937: 161). It is nowadays common to use the utility in modeling and comparing welfare.

34 Samuelson (1937: 159). This reservation does not affect the dominant role of the discounted utility. Almost all the papers involved with intertemporal decisions in well-cited theoretical economic journals use the utility concept.

35 Frederick *et al.* (2002).

36 Warner and Pleeter (2001). The studies, for instance, by Rader (1981), also hold that there is no reason to believe that different consumers have identical time preferences for utility streams. It should be remarked that Becker (1992) first observed that if individuals have heterogeneous constant rates of impatience, the representative agent will not in general use a constant rate to discount the future.

37 Frederick *et al.* (2002: 393–384).

38 As rational economics predicts, the acceptance of an alternative approach to a dominant idea like the discounted utility in a "sophisticated" game field like theoretical economics may take years or decades. Keynes (1936) says: "the difficulty lies, not in the new ideas, but in escaping the old ones, which ramify … into every corner of our minds."

39 I was aware of many fundamental questions in the Ramsey approach when I was writing graduate thesis on applying Lie group theory to the housing market in 1985 at the Department of Civil Engineering, Kyoto University.

40 A comprehensive survey on time discounting and time preference is given by Frederick *et al.* (2002). It is observed that there is a growing list of anomalies for the discounted utility model – patterns of choice that are inconsistent with the model's theoretical predictions. Scholars have also proposed different models to replace the discounted utility model; but no one has been successful in the replacement. As observed by Frederick *et al.*, when this model

> eventually became entrenched as the dominant theoretical framework for modeling intertemporal framework for modeling intertemporal choice, it was due largely to

its simplicity and its resemblance to the familiar compound interest formula, and not as a result of empirical research demonstrating validity.

This book is based on an alternative utility approach to handle with intertemporal choice problems without reference to the compound interest formula.

41 A recent refinement and extensions of the approach for national economies are given by Zhang (2005a). The theory provides an integrated approach to theory of national economic growth and development (Zhang, 1999), international trade theory (Zhang, 2000), urban economics (Zhang, 2002a), and interregional economics (Zhang, 2003a) with heterogeneous households and economic structure over time.

42 It would be helpful for the reader to read some books on monetary economics, for instance, by Champ and Freeman (2001), Walsh (2003) and Handa (2005).

43 This section provides a typical way in changing the continuous version to the discrete version. All the models in this book are developed in continuous time. Zhang (2005a: Section 4.8) introduces money into the OSG model in discrete time. The reader is encouraged to re-examine the other models in continuous time in this book in discrete version. For mathematical reference, see Zhang's recent book which deals with the modern theory of discrete dynamical systems and its applications to economics (Zhang, 2006b).

44 In principle, it may be many periods. Nevertheless, if a man lives more than two periods, complicated analytical problems will arise and the model may become analytically intractable.

45 The model was initially proposed by Samuelson (1958) and Diamond (1965). The model here is based on Blanchard (1997) and Section 3.8 in Barro and Sala-i-Martin (2005). Some extensions of the model are referred to by de la Croix and Michel (2002). For extensions of the model, see also Azariadis (1993) and Azariadis and Lambertini (2003).

46 As mentioned in Azaridias (1993), each period should last over 30 years if one really wants to use analytical results to provide direct insights into reality. The length of this period is generally considered too long for discussing changes, for instance, in prices, government policy, knowledge, capital, or unemployment. Nevertheless, theoretical economists have accepted the framework mainly because it makes many growth problems analytically tractable.

2 Money as a store of value

1 Outside money is the part of money stock which is issued by the government.

2 The increase per unit of real money balance is given by

$$\frac{1}{M/P}\frac{d(M/P)}{dt} = \mu - \pi.$$

3 As mentioned in the next section, the well-known Sidrauski model predicts superneutrality.

4 The model developed in this section with money acting as single outside asset is influenced by the OLG models in Azariadis (1993), Tirole (1985), and von Thadden (1999).

5 See Stein (1970), Grandmont (1985, 1998), and Chirichiello (2000).

6 The assumption that the government's role is only to keep the money growth rate at a constant rate and to distribute seigniorage to the representative household as a transfer payment in a lump-sum manner is a well-accepted assumption in the literature on money and growth (for instance, Orphanides and Solow (1990), Marquist and Reffett (1995) and Mino and Shibata (1995)).

7 Monetary models in which the demand for money is purely speculative have been studied by Shell *et al.* (1969) and Hahn (1969).
8 Here, we consider that the conduct of monetary policy takes precedence over all fiscal matters. Gale (1983: Chapter 2) provides a discussion of how to resolve the coordination problem between monetary and fiscal measures in a similar context.
9 The substitution of capital for fiat money in reaction to an increase in anticipated inflation is called the Tobin effect, as described in Tobin (1965). See also Champ and Freeman (2001). An analysis similar to Tobin's is due to Mundell (1965). Hence, the literature uses the terms "Tobin effect" and "Mundell-Tobin effect" interchangeably.
10 It should be noted that changes in the parameter will affect paths of the consumption and total wealth towards their respective equilibrium values.
11 Villanueva's model predicts possibility of multiple equilibrium points in the monetary economy.
12 Some implications of this relationship for the forward-looking behavior of bond prices are referred to Turnovsky (2000: Chapters 7 and 8).
13 It is also important to examine other policy instruments, as Turnovsky (2000) did. We leave these questions to the reader as exercises.
14 It should be remarked $r(t) = i(t) - \pi(t)$.
15 As $\mu_0 \leq 1/2$ and $s' < 1$, for sufficiently large m we have $s(k_0 + \mu_0 m) < m/2$.
16 Weil (1987, 1991) show that if there are different types of households, changes in government monetary policy should have redistributional effects and will affect capital accumulation. Mino and Shibata (1995) also examine redistributional issues in a monetary growth model in the money in utility with the production technology proposed by Jones and Manuelli (1990).
17 Atkinson and Bourguignon (2000: 5).
18 The government may distribute the inflation tax according to each group's relative as well as absolute economic conditions. That is, μ_j, are endogenous variables.
19 $M_j = \mu_j M/\mu$ implies $\dot{M}_j/P = \mu_j \dot{M}/\mu P = \mu_j m$. As $\mu_j m$ is the transfer from the government and $\dot{M}_j/P$ is the change of household j's real money holdings, $M_j(t) = \mu_j M(t)/\mu$ is guaranteed if $M_j(0) = \mu_j M(0)/\mu$ and no household will transfer real money balances to physical capital. This assumption is acceptable as the households are indifferent to holding money and physical wealth.
20 We are only interested in the case of $m > 0$. The case of $m = 0$ is examined in Zhang (2006a).
21 The following term is positive in the two-sector model without money (Zhang, 2006a).
22 It should be noted that even if this requirement is not satisfied, it is expected that the steady state is a saddle point. Nevertheless, as the expressions are tedious, it is difficult to explicitly determine the signs of the parameters.
23 See, for instance, Drandakis (1963), Diamond (1965b), Weizsäcker (1966),Corden (1966), Stiglitz (1967), Gram (1976), Mino (1996), Benhabib *et al.* (2000) and Drugeon and Venditti (2001).
24 If $R(t)$ and $W(t)$ are respectively nominal rate of interest and nominal wage rate, then $r(t) = R(t)/P_i(t)$ and $w(t) = W(t)/P_i(t)$.
25 We neglect the case of $m = 0$, which is fully examined in Zhang (2006a: Chapter 5).
26 As it is difficult to determine whether the steady state solution is meaningful, we will discuss the issues later with some specified production functions.
27 The reader refers to Zhang (2005a: Chapter 6; 2005b) for the two-sector economy without money.
28 See Azariadis (1993), Galor and Ryder (1989), and von Thadden (1999).
29 See Tobin (1955, 1965). Some extensions of this model to the Keynesian economics are referred to Chiarella and Flaschel (2000) and Asada (2007). Franke and Asada (1994) construct a nonlinear dynamic IS-LM model in four state variables representing real

balances, inflation, income distribution, and a so-called state of confidence. The system exhibits time-dependent behavior.

30 The model and analysis in this section are based on Jullien (1988). A model in the same spirit is proposed by Grandmont (1985), even though Grandmont's model does not take account of capital accumulation. See also Farmer (1986). As far as technical issues related to existence of chaos are concerned, the reader is also referred to Yokoo (2000).

31 A non-monetary steady state k verifies $s(W(k), f'(k)) = k$. It is inefficient if $k > k^*$, where $s(W(k^*), 1) > k^*$. For sufficiently large k, $W(k) < k$ Hence, for sufficiently large k, we have $s(W(k), f'(k)) < W(k) < k$. As s is continuous and it maps itself, it has at least one fixed point.

3 Money in utility and production functions

1 See Eden (2005: Chapter 2) for the reasons why money is introduced into the utility function.

2 Superneutrality of money means that the growth rate of money has no effect on the real equilibrium.

3 It should be noted that most variables are defined the same as in the Tobin model in Chapter 2.

4 We refer further explanations of this definition to Levhari and Patinkin's 1969 paper.

5 This version of the Sidrauski model is based on Orphanides and Solow (1990: Section 2.1). See Fischer (1979a, 1979b), Asako (1983) and Reis (2007) for analysis of the dynamic properties of the model.

6 Unless otherwise specified the notation is the same as in Section 3.1.

7 Although inflation does affect the demand for real balances, it affects welfare since money is an argument in the utility function. Further comparison between the Sidrauski model and the Tobin model is referred to in Orphanides and Solow (1990).

8 For instance, Barro (1995) finds that inflation has a negative effect on growth and MaCandless and Weber (1995) find no relationship between inflation and average real growth.

9 Brock (1974) introduces leisure time explicitly into the Sidrauski model. As mentioned by Brock (1974: 752), "Endogenous labor supply enables us to see what special assumptions (viz., separability of consumption, real balances and leisure) are needed to produce the long run neutrality result." The version of the model here is based on Turnovsky (2000: Section 8.8). See also Brock and Turnovsky (1981), Turnovsky (1987) and Gomme (1993). As far as money in utility is concerned, another two interesting models are built by Marini and van der Ploeg (1988) and van der Ploeg and Alogoskoufis (1994).

10 See Fischer (1979a) and Turnovsky (1987).

11 This is a popular way of introducing externality in the recent literature of economic growth. An interpretation of the term, $\bar{K}^{\psi}(t)$, is to consider it as a measurement of knowledge, which is accumulated through Arrow's learning and by doing.

12 In this model, nominal money growth is a means of public finance.

13 It should be noted that Barnett (2005) deals with a similar problem of coordinating macroeconomic policy, using a simple AK model.

14 It should be remarked that this book assumes that as the cost of inflations are related to the positive opportunity costs of holding money, an alternative to deflation as a means of eliminating these costs would be the payment of explicit interest on money (see, for instance, Walsh, 2003). If we assume that the government pays a nominal interest rate of i^m on money balances, then the budget constraint of the household becomes

$$\hat{y} = rk + w + (i^i - \pi)m + \tau + a,$$

The government's budget constraint is correspondingly changed.

15 Here the specified functional form already implies the assumption that real balances and consumption are Edgeworth complements ($u_{cm} > 0$). If the assumption is replaced with the assumption of Edgeworth substitutability ($u_{cm} < 0$), then the dynamic properties may be affected.

16 It should be noted that as observed by Orphanides and Solow (1990: Section 2.1), if the utility function in the Sidrauski model is specified as $U = \log(c) + \log(m)$, then $(r(t) + \pi(t))m(t) = c(t)$, a relation similarly held in our approach. Another popular hypothesis about the temporal utility function is $U = u(c) + v(m)$, where the sub-utility function, $v(m)$, is assumed to be twice-continuously differentiable, and the marginal utility of money satisfies: $\text{sgn}[v'(m)] = \text{sgn}\lfloor m^* - m \rfloor$, where m^* is the satiation level of real money balances (see Friedman, 1970).

17 For instance, when $f = Ak^{\alpha}$, if $\alpha < 1/2$ the requirements are satisfied.

18 See Zhang (2005a: Section 2.6).

19 The government may distribute the inflation tax according to each group's relative as well as absolute economic conditions. That is, μ_j, are endogenous variables, rather than parameters.

20 See Arrow (1962), Uzawa (1965), Kennedy (1964), Weisäcker (1966), Samuelson (1965) and Schultz (1981).

21 Works, for instance, by Sato and Tsutsui (1984) and Nelson and Winter (1982), are examples along this line.

22 Romer (1986).

23 See, for example, Dollar (1986), Krugman (1991), Rauch (1991), Stokey (1991), Nardini (2001).

24 There is a large number of literature on economic growth with bifurcations and chaos (for instance, Day (1984), Hommes (1991, 1998), Zhang (1990, 1991), Azariadis (1993), Boldrin and Woodford (1990), Boldrin, *et al.* (2001), Matsuyama (1991, 2001), Shone (2002)). Zhang (2005c, 2006b) systematically introduces contemporary theories of differential and difference equations and their applications to economics.

25 Wang and Yip (1992) introduce money into a growth model with education sector by assuming that money directly affects production functions of the goods sector. Their model results in money superneutrality. In a similar modeling framework but introducing physical capital into the production of human capital, Pecorino (1995) shows that money is not neutral. See also Bond *et al.* (1996) and Chang (2002) for introducing money into growth models with human capital accumulation.

26 Arrow (1962) and Uzawa (1965). Learning by doing has been introduced into growth theory in different ways. For instance, Chari and Hopenhayn (1991), Parente (1994), and Stokey (1988) study learning by doing as a force for sustained growth; Brezis *et al.* (1993), Krusell and Rios-Rull (1996), and Jovanovic and Nyarko (1996) show that learning by doing can give rise to the overtaking; Karp and Lee (2001) examined learning by doing and the choice of technology; and Liso *et al.* (2002) examine the implications of learning by doing for division of labor.

27 In a two-sector growth model with endogenous human capital by Chang (2002), the production of human capital is also dependent on money. For simplicity of analysis, we do not introduce money into the human capital accumulation equation. The common sense tells us that modern education needs money, just like services and goods sectors.

28 Here, we assume $a_1^2 - 4a_2 \geq 0$, which is difficult to explicitly judge.

29 Although it is important to examine dynamic properties of the monetary economy, as the behavior becomes too complicated, we simulate the nonmonetary economy.

30 This case is examined in Zhang (2006a: Section 3.2). The procedures to determine the dynamics and steady states are referred to by Zhang.

31 The modern economic developments of Confucian regions, Japan, Korea, Singapore, Hong Kong, and mainland China are systematically examined by Zhang (1998a, 2002b, 2003b, 2006c and 2007a).
32 Among well-known East Asian thinkers, perhaps only Fukuzawa Yukichi (1835–1901) is exceptional.
33 The assumption that money yields direct utility guarantees that money will be valued if the utility function is properly defined.
34 For instance, Imrohproğlu (1992) studies interdependence between income fluctuations and holding money. Feenstra (1986) provides some possible transactions stories, which can be represented by entering money in the utility function.
35 The assumption that money affects production possibilities is accepted in many studies, for instance, by Fischer (1974), Taylor (1977), Calvo (1979) and Benhabib *et al.* (2001a, 2001b).
36 See, for instance, Simos (1981), You (1981), and Odedokum (1999).
37 For illustration, we do not consider the possibility that money enters the utility function.
38 Becker (1965). Becker's works are mainly static. There are many growth models with leisure as an endogenous variable. For instance, Barro and Sala-I-Martin (1995) propose a one-sector growth model with endogenous time within the Ramsey framework. Nevertheless, as demonstrated by Barro and Sala-I-Martin, a growth model even with simple utility and production functions becomes complicated. A comprehensive literature review on time in economic theory is provided by Zamagni and Agliardi (2004).
39 See, for instance, Benhabib and Perli (1994), Ladrón-de-Guevara *et al.* (1997), Turnovsky (1999), Greenwood and Hercowitz (1991), Rupert *et al.* (2000), and Campbell and Ludvigson (2001).
40 See also Dowd (1990).
41 See Iooss and Joseph (1980), Hale and Koçak (1991), Kuznetsov (1998), and Zhang (2005b).
42 This section is based on Matsuyama (1991).
43 Strictly speaking, $m(t) = 0$ or $P(t) = \infty$ is another candidate for the steady state equilibrium. Nevertheless, for the utility function used below, this case is ruled out.
44 Under this requirement, the transversality condition $\lim_{t\to\infty} \beta^t p^{\eta}(t) = 0$ is automatically satisfied for $p \in (0, 1)$.
45 See Coppel (1955).
46 This model is a straightforward extension of Zhang's growth model of heterogeneous households by introducing money (Zhang, 2006a: Section 4.2).
47 In the general equilibrium theory, the number of households is equal to the number of types of households. It can be seen that the analytical framework proposed in this book can include the general equilibrium theory, as far as economic relations among variables are concerned, as a special case where capital and other stock variables are not changeable (or in stationary state).
48 We do not provide explicit expressions as they are too complicated. Moreover, we will not use them as we are only concerned with equilibrium.
49 The recent literature on Ramsey model with endogenous leisure and money is referred to, for instance, Dutkowsky and Foote (1996), Dutkowsky and Dunsky (1996), Turnovsky (2000: Section 8.8) and Ho *et al.* (2007).
50 The literature of growth theory with endogenous fertility is referred to, for instance, Barro and Becker (1989), Razin *et al.* (1995), Nerlove and Raut (1997), and Yip and Zhang (1997).
51 The approach is based on Wang *et al.* (1994) and Palivos (1995). An alternative approach is to consider explicitly a child-bearing function (which is dependent on n and k) in the budget constraint of the representative consumer. As child-bearing costs both time and money, a proper approach should synthesize the two approaches.
52 When $T_c'' < 0$, there are economies of scale in child-bearing (Palivos, 1995).

4 Money-in-advance approaches

1 The MIU approach has not been replaced by this approach partly because the introduction of the various constraints associated with money transactions tends to result in models, which are analytically intractable.
2 An early formal explanation between cash holdings and rate of interest is given by Baumol (1952). The discussion below about the Baumol's equation is based on Pentecost (2000: Chapter 11). See Baumol (1952) for further explanations about the economic implications of the model.
3 See also other models by, for instance, Tobin (1956), Miller and Orr (1966), Saving (1972), and Abel (1985) for deriving demand functions for money under different conditions.
4 In this section, we only consider money for purchasing consumption goods and neglect possible money required for other purposes, such as investment.
5 According to Stockman, investment should also be taken into account. That is, instead of (4.1.3), one should have a general constraint in the form of $\Psi(c, \dot{k}) \leq m$. A simplified form is $c + \dot{k} \leq m$.It should be remarked that the form accepted in this study is quite common in the CIA literature and it is also important for us to generate a unique steady state. If we accept $\Psi(c, \dot{k}) \leq m$, then the uniqueness may not be guaranteed. Suen and Yip (2005) discuss how CIA constraints may affect the number of steady states and their stability properties in the Ramsey approach.
6 Zhang (2006a: Section 5.2) develops a two-sector growth model, basing on Uzawa's two-sector model.
7 If $R(t)$ and $W(t)$ are respectively nominal rate of interest and nominal wage rate, then $r(t) = R(t)/P_i(t)$ and $w(t) = W(t)/P_i(t)$.
8 In this section, we only consider money for purchasing consumption goods and neglect possible money required for other purposes, such as investment. To make the analysis more realistic, for instance, we consider that the total demand for money is divided into the households' demand, $M_c(t)$, the capital sector's demand, $M_i(t)$, and the consumption sector's demand, $M_s(t)$. Assuming that $M_j = \chi_j P_j F_j$, $j = i, s$, we can derive the total demand function for money.
9 We neglect the case when the parameter is equal to zero, as the model of the case is fully examined by Zhang (2006a).
10 We neglect the case of $m = 0$, which is fully examined in Zhang (2006: Chapter 5).
11 A similar economy with the traditional Ramsey approach is modeled by Meng (2002). As another exercise of our approach, the reader is encouraged to introduce capital with forward-interest rule in the way as suggested by Huang and Meng (2007). It should be noted that we are not concerned with issues related to the length of time that money is held. Romer (1986) proposes a model in which the money-holding period is an endogenous decision. The model is based on the Baumol-Tobin model – cash withdrawing costs and agents must pay for their consumption purchases with cash.
12 It is possible to reasonably make the parameters endogenous in such a way that the labor supply curve is S-shaped and the system has multiple equilibria. See, for example, Dessing (2002, 2004) and Wetzell (2002) for further examinations. Combing the model in this section and the model with endogenous leisure in Section 4.2, we can examine relations between government spending and capital accumulation with endogenous time preference – the topic is addressed by Chang *et al.* (1998a, 1998b) with the Ramsey approach.
13 It can be shown that the following equation can also be expressed as follows

$$\dot{a}(t) = T(t) + (R_b(t) - \pi(t))a(t) - \tau(t) - c(t) - R_b(t)m(t).$$

This equation is identical to the budget constraint in Meng (2002).
14 See Taylor (1993).

15 Some recent empirical studies on the issues related to the role of modern monetary policy are referred to Hafer *et al.* (2007).
16 This specification implies that the policy maker sets the nominal interest rate as a function of an instantaneous rate of inflation. This is not a realistic assumption as instantaneous measures of inflation do not exist. In the literature of monetary economics inflation measure to which the central bank responds is related to some average of past inflation rate of forecasts of expected future of inflation, or both (for instance, Orphanides (1997), Krugman *et al.* (1985), Leeper (1991), Clarida *et al.* (1998), Svensson (1997), Benhabib *et al.* (2001a, 2002a)).
17 The following relation can also be obtained by the goods market clearing condition, $c = 1 - T_h$, and equations (4.3.4).
18 As Meng (2002) tries to show by the traditional approach that when R_b takes on different forms, the system may have either unique or multiple steady states.
19 Benhabib and Uribe (2002) demonstrate multiple steady states with different forms of the function. Following Benhabib *et al.* (2002b), one may also discuss issues related to liquidity traps within the modeling framework. See also Benhabib (2004) for other types of monetary policy.
20 It is also an interesting problem, following the ideas suggested in Benhabib *et al.* (2002b), to generate chaos by developing the model in this section in discrete version. See also the recent studies by Carlstrom and Fuerst (2005) and Sveen and Weinke (2005).
21 See Rosenzweig and Stark (1997), Gray (1998), Chiappori *et al.* (2002) and Suen *et al.* (2003) for further literature on the subject. It should be noted that lifetime is also an endogenous variable in the recent literature (e.g. Ehrilich and Lui (1991) and Chakraborty (2004)).
22 The model is a simplified version of the growth model with housing and location proposed by Zhang (1996). As pointed out by Greenwood and Hercowitz (1991), the stock of household capital actually exceeds market capital. Benhabib *et al.* (1991) estimate that the output of the household sector may be as much as half that of the market sector and that labor hours in the home sector are almost as great as in the market sector. Hence, proper introduction of household capital in growth theory is essential. We refer monetary growth models with durable consumption goods to, for instance, Mohsin (2006) and Erceg and Levin (2006).
23 That is, $\eta_0 > \alpha\xi_0$.
24 It is straightforward to show that if

$$\left(\xi - \frac{\lambda}{\alpha(1+\chi\mu)}\right)\left(\frac{1}{\chi}+\mu\right) + \lambda\delta_k\left(\frac{1}{\chi}+\mu-\frac{1}{\lambda}\right) > (<)0,$$

then

$$\left(\frac{\lambda\alpha A}{\xi\bar{\mu}+\lambda\delta_k}\right)^{1/\beta_i} > (<)\left[\frac{(\bar{\chi}-1)\alpha A}{\delta_k\bar{\chi}}\right]^{1/\beta_i}.$$

25 The nonmonetary model in this section is referred to Zhang (2006a: Section 3.3.).
26 Banerjee and Newman (1993), for instance, address issues related to development and occupational choice by focusing on the interplay between agents' occupational decisions and the distribution of wealth. Their study is restricted to capital market imperfections and the situation in which poor agents choose working for a wage over self-employment and wealthy agents' become entrepreneurs who monitor workers.
27 Marquist and Reffet (1991) and Mino and Shibata (1995) also introduce money into two-sector models involving human capital via a cash-in-advance constraint.
28 In Zhang (2006a: Section 3.3), it is assumed that the two professions offer different professional amenities. The introduction of amenity differences into the dynamic model

will make the assumption that wage rates are professionally equal invalid. For simplicity of analysis, this section assumes that the two professions offer the same level of professional amenities. As professional choice is free and costless, the wage rates would be equal.

29 Suen and Yip (2005) discuss how CIA constraints may affect the number of steady states and their stability properties in the Ramsey approach.

30 This is satisfied if μ is small.

31 In a two-sector growth model with human capital, Marquist and Reffet (1991) show that an increase in the rate of nominal money supply depresses the long-term economic growth, as long as the cash-in-advance constraint applies to investment demand for either physical or human capital.

32 See Barro (1991). There are numerous models on growth with government spending. Barro (1990) builds a growth model with government spending and income taxation where government activity enters directly into production function as a public intermediate input. King and Rebelo (1990) compute the growth and welfare effects of income taxes in a two-sector growth model. Rebel (1991) develops a number of basic endogenous growth models and explore the impact of various government policies on balanced growth paths. Glomm and Ravikumar (1994, 1997) examine growth and government's investments. Davis (2003) examines the relations between growth of government and the division of labor. See also Shleifer and Vishny (1993), Tataru (1999), Krusell and Rios-Rull (1999), Song (2002), Barreto (2000), Waller *et al.* (2002) and Dixit (2003) for possible directions of extensions of our model.

33 This model is examined in detail in Chapter 4 in Zhang (2005a). A model, which discusses similar issues as to this one but is developed within the traditional optimal growth framework, is proposed by Guo (2004). See also Devereux *et al.* (1996, 2000), Chang *et al.* (1998a, 1998b), Bajo-Rubio (2000), Zhang (2000a), and Chatterjee *et al.* (2003).

34 Different ways of taxation in different analytical frameworks are referred to Jha (1998).

35 See Zhang (2005a: Section 4.4).

36 In the figure, δ^*, ρ and n are positive parameters.

37 The model is referred to Orphanides and Solow (1990: Section 4.1). See also Jha *et al.* (2002). Croushore (1993) demonstrates the functional equivalence between the shopping-time model and the money in the utility function formulation.

38 It can be seen that this model is the same as the model of money in the production function, if we assume that production function as $f = (1 - \varphi(m))f(k)$. See also Benhabib and Bull (1983), Zhang (2000a) and Suen and Yip (2005) for introducing transactions costs in production into the Ramsey framework.

39 See, for instance, McCallum (1983) and Kimbrough (1986).

40 That money is held primarily to facilitate savings in time also finds empirical support in the works by Dowd (1990).

41 In Love and Wen (1999), θ_0, is a function of consumption tax rate, τ_c, in the form of $a(1+\tau_c)^{\theta}$. For simplicity, this study does not examine effects of tax structure.

42 This case is analyzed by Zhang (2005a: Section 4.2).

5 Unemployment and money

1 Appendix A.1 introduces a general model to explain the relations between the inflation and rate of unemployment. The model demonstrates the existence of chaos.

2 The model and its solution are based on Chiang (1985: 435–538) and Zhang (2006a: 38–40). See also Stein (1974) and Order (1977).

3 We specify: $a = 0.9, b = 0.1, h = 0.3, m = 2.9, j = 1$ and $k = 0.6$.

4 In his 1937 paper, Hicks actually tried to provide a graphical representation of the ideas of Keynes' *The General Theory of Employment, Interest, and Money*.

5 The IS-LM versions in this and the next sections are based on Abel and Bernanke (1998: Chapter 10). See also Jha (2003: Chapter 1). It should be pointed out that Bénassy (2007) constructs a dynamic general equilibrium which displays the central features of the IS-LM model.
6 See, for instance, Hahn and Solow (1985).
7 See Abel and Bernanke (1998: Chapter 10).
8 The table is based on Blanchard (1997: 419).
9 Another alternative approach to modeling the problem is to distribute the government's inflation income equally among the population and the unemployment is paid by the government's tax income.
10 We will not discuss conditions for existence of meaningful solutions because the expressions are tedious.
11 As can be seen from discussions below, this requirement is only for convenience of discussion as we have already solved the equilibrium.
12 If $\delta_2 > 0$, we should discuss the condition for $\delta_0^2 \geq 4\delta_1$. We will not discuss this situation as the expressions are tedious.
13 Jellah and Zenou (2000). It is also interesting to see another approach to employment dynamics with endogenous preferences proposed by Pemberton (1985).
14 Effort level should be dependent not only on the wage rate but also on wealth and other factors such as educational level. In Zhang (2005a: Chapter 8), the case that the efficiency is a function of the real wage rate and wealth-worker ratio is examined.
15 We do not discuss conditions for existence of meaningful solutions because the expressions are tedious.
16 There is a large number of literature on the subject. See, for instance, Clower (1965), Bénassy (1975, 1982), Leijonhufvud (1968), Malinvaud (1977, 1998), Hahn (1978), Green and Laffont (1981), Dréze (1975), and Negishi (1985). For the recent developments, see Zhang (1999), Chiarella and Flaschel (2000) and Bignami *et al.* (2004).
17 An introduction to the theory is provided by Pentecost (2000).
18 Neugart (2004) builds a labor market model which exhibits chaos in unemployment.
19 How labor unions play games is discussed economically by Booth (2002). Further issues related to growth and unemployment are referred to by Zhang (1993a), Aghion and Howitt (1994), and Sener (2001).
20 This model is based on Neugart (2004). The mathematical terms and theorems are referred to by Zhang (2006b: Section 7.6).

6 Preference change and habit formation

1 Unless otherwise specified the notation is the same as in Section 3.1.
2 There are numerous models with endogenous change of time preference, for example, Uzawa (1968), Wan (1970), Boyer (1978), Shi and Epstein (1993), Lahiri and Puhakka (1998), Faria (2002), and Das (2003).
3 Blanchard and Fischer (1989: 74–5).
4 Turnovsky (2000: 357).
5 We refer to the stability analysis in Section 4.2 when the preference is fixed and to Zhang (2005a: Section 11.1) when money does not explicitly affect the economic system.
6 Here the specified functional form already implies the assumption that real balances and consumption are Edgeworth complements ($u_{cm} > 0$). If the assumption is replaced with the assumption of Edgeworth substitutability ($u_{cm} < 0$), then the dynamic properties may be affected. Benhabib *et al.* (2001b) show how these two assumptions lead to different dynamic properties of monetary economies in the Ramsey approach.
7 If the propensity to hold real money balance is equal to zero, the economy becomes non-monetary.

8 If the preference change is dependent on the income and wealth distribution, then our approach is related to the literature of economic growth with social status (see, Zhang, 2006a).
9 Under proper assumptions, the preference can also be related to the past preference and living conditions. It is possible to relate our approach to the literature of economic growth with habit persistence.
10 See, for instance, Constantinides (1990), Abel (1990), Mansoorian (1996, 1998), Carroll *et al.* (2000), Fuhrer (2000), Díaz *et al.* (2003), Alonso-Carrera *et al.* (2005), and Gómez (2006). As mentioned in Faria (2001), some empirical studies support significance of the hypothesis of habit formation, and other studies do not find evidence for supporting the hypothesis.
11 This is a strict assumption because habits should develop over past levels of both consumption of goods and real money balances. The well-being of the representative household depends not only on its past consumption, but also on how smoothly this consumption was obtained through transactions.
12 In this model, money is superneutral as in the Sidrauski model, even though the demand for money in this model is larger than in the Sidrauski model; and the Friedman rule (which says that the steady utility can be maximized by making real money balances large enough that their marginal utility equals zero, that is, $r = -\pi$). The money demand reflects the habit persistence related to past consumption and the rate of time preference, and it is more sensitive to changes in inflation and nominal interest rates than the Sidrauski model. See also Bouakez *et al.* (2005), Christiano *et al.* (2005), and Lioui and Rangvid (2007) for the recent literature related to habit and money in growth model.

7 Monetary growth with urban structure

1 We refer two comprehensive surveys on the literature to Henderson and Thisse (2004) and Capello and Nijkamp (2004).
2 For instance, Lucas (1988).
3 See, for instance, Richardson (1973), Rabenau (1979), and Henderson (1985). As mentioned later on, a few theoretical models are proposed to deal with urban growth with economic geography. Nevertheless, these approaches are different from the model in this chapter in that this study is based on the alternative approach to household behavior.
4 See, for instance, DiPasquale and Wheaton (1996) and Silos (2006).
5 See, for instance, Muth (1973), Anas (1978, 1992), Hockman and Pines (1980a, 1980b), Brueckner (1981), Brueckner and Pereira (1994), Arnott (1987), Arnott *et al.* (1999), and Braid (2001).
6 More than one income class was introduced by Beckman (1969) and Solow (1973) among others. See also Beckmann and Papageorgiou (1989), Anas (1990) and Tabuchi and Thisse (2002) for some other urban models with heterogeneous households.
7 Our approach is mainly based on Anas (1978), even though the Anas housing production also includes the labor input. Further issues related to the durability of real estates and its costly conversion and replacements are also discussed by Anas. See also Arnott (1980), Arnott *et al.* (1999) and Glaeser and Gyourko (2005) for introducing more realistic aspects of the housing market to the growth model. A recent literature review is referred to by Lin *et al.* (2004).
8 Another two popular assumptions in the literature of urban economics are the absentee landownership and the public ownership. In the former, land is owned by absentee landlords who spend their land incomes outside the economic system. In the latter, for instance as accepted in Kanemoto (1980), the city government rents the land from the landowners at certain rent and sublets it to households at the market rent, using the net revenue to subsidize city residents equally.

9 In modern cities, the land at a special location – such as subways and railways passing through buildings – may be used for business, housing and transportation.
10 This study does not take account of externalities for producers. For instance, firms often prefer to locate to other firms. An explicit introduction of externalities will make the spatial structure far more complicated. In the literature of urban economics, various externalities have been analyzed (see, Henderson (1974), Upton (1981) and Abdel-Rahman (2004)).
11 Their empirical study demonstrates that high amenity cities grow faster than low amenity cities and that the role of urban density in maintaining urban growth is important.
12 This specified form is a limited case. Locational amenities or disamenities are not only affected by the residential density at the location. For instance, possible social contacts of any individual are spread over the whole space. Air pollution is not limited to locals.
13 It can be shown that this equation is redundant in the sense that it can be derived from the other equations in the model. This also follows from Walras' law.
14 It should be emphasized that we only assume the equality in utility level but not income and wealth in the model building. It can be seen that if we introduce transportation costs or/and the absentee landownership, much more analytical difficulties may arise.
15 This implies, for instance, that the travel time, $\Gamma(\omega)$, between the CBD and the residence is reduced as time passes.
16 Substituting equation (7.1.2) into (7.1.9), we have $U = \theta T_h^{\sigma} c^{\xi} k_h^{\eta\alpha_h} L_h^{\eta\beta_h} s^{\lambda}$. Hence, we may interpret $\eta\beta_h$ as the propensity to use land.
17 In this model, we assume $T_h(L) > 0$.If all the available time is spent on traveling, the residents of the other end of the isolated state have some leisure time.
18 Another way to take account of social interactions is proposed by Papageorgiou and Smith (1983). They consider a trade-off between social contacts and land use. The need for social contacts is negatively related to distance and the need for land is negatively affected by crowding. When the propensity to interact with others (which corresponds to μ in our approach) is large enough, the urban equilibrium configuration becomes unstable.
19 This result is also explained in the Muth model (see Fujita, 1989: Section3.7).
20 Some empirical studies on the US economy demonstrate that the value of the parameter, α, in the Cobb-Douglas production is approximately equal to 0.3 (for instance, Abel and Bernanke, 1998).
21 As the parameters will not affect the national economic production and income, we do not mention the location-independent variables in the rest of this section.
22 This conclusion hints at possible impacts of the family structure and sexual division of labor on the economic geography. For instance, according to this theory, in Japan where husbands travel to work and wives spend time on shopping and local communications, the urban configuration tends to be flat. Zhang (2005a) develops growth models with sexual division of labor. It is straightforward to extend this model to take account of issues related to family structure, economic geography, and growth.
23 Zhang (2005a, 2006a) have extended and generalized the neoclassical growth theory in many directions with the utility function used in this study. For contemporary literature on spatial economics, see Fujita and Thisse (2002b), Forslid and Ottaviano (2003), Henderson and Thisse (2004) and Robert-Nicound (2004).
24 See also Imai (1982), Lucas and Rossi-Hansberg (2002), and Berliant *et al.* (2002), and Rossi-Hansberg and Wright (2007) for this direction of research.
25 See, for instance, Turnovsky and Okuyama (1994), Skinner (1996), and Burbidge and Cuff (2005).
26 It can be shown that this equation is redundant in the sense that it can be derived from the other equations in the model. This also follows from Walras' law.
27 That is, $k(\omega, 0) = K(0)/N$.
28 That is, $k(\omega, 0) = K(0)/N$.

8 Money in multi-regional and growth economies

1 For instance, Krugman (1991), Fujita *et al.* (1999), Forslid and Ottaviano (2003) and Yamamoto (2005).
2 See, for instance, Hanson (2001).
3 Zhang proposes different interregional models, also including knowledge and human capital accumulation and other aspects of interregional economic dynamics (Zhang, 2003a, 2007b, 2007c, 2008b). This book is limited to only one model as the main purpose is to demonstrate how to introduce money into interregional dynamics.
4 See McDougall (2002).
5 The approach is represented, for instance, by Ianchovichina and McDougall (2001).
6 The literature of this approach is vast, for instance, Isard (1953), Boomsma and Oosterhaven (1992), Sonis *et al.* (2001) and Canning and Wang (2005).
7 Trade models with capital are referred to, for instance, Oniki and Uzawa (1965), Brecher *et al.* (2002), and Sorger (2003).
8 Transportation costs are introduced into interregional models, for instance, by Takayama and Judge (1971) and Roy and Johansson (1993).
9 Fujita and Thisse (2002b) provide a review of the literature.
10 Refer to, for instance, Glaeser and Maré (2001), Duranton and Monastiriotis (2002) and Rey and Janikas (2005).
11 For instance, Andersson and Forslid (2003), Baldwin and Krugman (2004), Bayindir-Upmann and Ziad (2005), Borck and Pflüger (2006) and Makris (2006).
12 Zhang, (2005a, 2006b, 2008a).

9 Money, growth, and international trade

1 This section is based on Dornbusch and Giovannini (2000: Section 2).
2 Further literature is referred to by Dornbusch and Giovannini (2000).
3 Frenkel and Razin (1987) pointed out that although they wrote independently, Mundell's contributions are more important than Fleming's.
4 See Abel and Bernanke (1998: Chapters 10 and 14), Dornbusch and Giovannini (2000), and Gandolfo (2001, Chapters 10 and 11).
5 The model is proposed by Dornbusch (1976), for an open economy adaptation of the IS-LM structure. The example is from Azariadis (1993: 46–51) and Zhang (2006a: 306–10).
6 See, for instance, Frenkel (1979), Wilson (1979), Gray and Turnovsky (1979), Mussa (1981), Buiter and Miller (1982) and Obstfeld and Rogoff (1984).
7 For instance Sachs (1981, 1982), Hodrick (1982) and Obstfeld and Stockman (1985).
8 See Turnovsky (2000: Chapter 11).
9 To solve this unrealistic assumption of the model, an alternative approach is to make the time preference an endogenous variable (Obstfeld, 1981), based on Uzawa's preference.
10 The environment we consider here is characteristic of those assumed in the standard monetary models of small open economies (see, for instance, Kouri (1974), Krugman (1979), Dornbusch and Fischer (1980), Obstfeld (1981), Obstfeld and Rogoff, (1995, 1998) and Gandolfo (2007)).
11 We assume that foreigners do not hold domestic money.
12 Generally, $i(t) = \hat{i}(t) + \varepsilon(t)$, where $\varepsilon(t)$ is expected rate of exchange depreciation. This condition is called uncovered interest rate parity. Under the assumption of perfect foresight, the expected rate of exchange depreciation is equal to the actual rate of exchange depreciation rate, $e(t)$.
13 It should be noted that this result depends on the assumptions that the output is fixed and no capital accumulation is allowed.

14 It is straightforward to show that equation (5.1.7) can be expressed as follows

$$c(t)+\dot{m}(t)+\dot{b}(t)=T(t)+(\tilde{i}(t)-\tilde{\pi}(t))b(t)-\tau(t)-\pi(t)m(t).$$

where we also use equations (9.4.1) and (9.4.2). This equation is identical to the budget constraint in the representative agent model for a small open economy in Turnovsky (2000: 352). We see that given the assumptions of PPP and UIP, the real rates of return on holding bonds and money are $(\tilde{i}-\tilde{\pi})$ and $-\pi=-(\tilde{\pi}+e)$.

15 As the total possible output is unit, this requirement is reasonable.

16 This section is concerned with the simple case of utility function with leisure. Different forms of utility functions with endogenous home production have been recently applied in economic dynamics (for example, Greenwood and Hercowitz (1991), Rupert *et al.* (2000), and Campbell and Ludvigson (2001)).

17 It is straightforward to show that equation (9.5.6) can be expressed as follows

$$\dot{a}_j(t)=f_j-c_j(t)-\pi_j(t)m_j(t)+r(t)b_j(t)+\tau_j(t).$$

This equation is identical to the budget constraint in the macroeconomic dynamic model with money by Ikeda and Ono (1992: 638). The optimal behavior of the household in their model is to maximize

$$U_j=\int_0^\infty \left\{\ln c_j(t)+\alpha_j \ln m_j(t)\right\}\exp(-\rho_j t)dt,$$

subject to constraint (9.5.7) and some other constraints.

18 Different issues related to (non-monetary) international trade are analyzed by Zhang (2000) with the same approach as used in this book.

10 Money and economic complexity

1 Zeeman (1977).

2 Kuhn (1977).

3 Zhang (1999: 402–3).

4 This approach has been applied to non-monetary growth theory (Zhang, 2005a), urban economics (Zhang, 2002a), international trade theory (Zhang, 2000, 2008a), interregional economics (2003a). Based on the approach, Zhang (2006a) also provides a comprehensive growth theory with income and wealth distribution.

5 For instance, we refer to Walsh (2003) to some empirical observations about interdependence among prices, monetary aggregates, nominal and real interests, and output.

6 See, for instance, Sargent and Wallace (1973), Fischer (1979a), Obstfeld (1985), Chang and Lai (2000) and Zhao (2007).

7 Chang and Lai (2000), for instance, provide examples about how to examine the announcement effects of anticipated monetary policy upon behavior of economic systems.

8 This continuous version of the Cagan model is based on Obstfeld and Rogoff (1998: 8.2).

9 In Cagan's study, a hyperinflation was a period during which the price level of goods in terms of money rises at a rate average of at least 50 percent per month. This corresponds to an annual inflation of almost 13,000 percent.

10 We refer the LSW model to Handa (2005: Section 16.2). The model is a synthesis of the Lucas supply function (Lucas, 1972, 1973) and the Sargent-Wallace model (1975, 1976).

11 Handa (2005: Section 16.2) studies the behavior and economic implications of the model.
12 For instance, Johnson (1969), Marty (1969) and Saving (1970, 1973) discussed effects of outside and inside money in static frameworks.
13 Walsh (2003: 4). It can be seen that our analytical framework allows us to introduce government monetary policies.
14 See, for instance, Feenstra (1976, 1980), Frenkel *et al.* (1991, 1996), Jha (2003) and Handa (2005).
15 Friedman's money supply rule is that the money supply should increase at a constant rate roughly equal to the long-run rate of output growth.
16 It simply means that all traded goods prices are set in US dollars so that exchange rate pass-through into the US price level is zero.
17 Some studies examine effects of transactions changes, for instance, by Grandmont and Laroque (1975), Kiyotaki and Wright (1989), Cole and Stockman (1992), Ireland (1994a, 1994b) and Espinosa-Vega and Yip (1999).
18 See, for instance, Azam (1983). Another direction for extension is to consider money in mixed economic systems, see, for instance, Asada (1997).
19 See, for instance, Palivos and Yip (1995), Leith and Wren-Lewis (2000), Heijdra and Ligthart (2000), Rankin and Roffia (2003) and Ganelli (2007).

References

Abdel-Rahman, H.M. (2004) 'The City System Paradigm: New Frontiers', *Urban Dynamics and Growth: Advances in Urban Economics*, edited by Capello, R. and Nijkamp, P., Elsevier: Amsterdam.

Abel, A.B. (1985) 'Dynamic Behavior of Capital Accumulation in a Cash-in-Advance Model', *Journal of Monetary Economics*, 16, 55–71.

Abel, A.B. (1990) 'Asset Prices under Habit Formation and Catching up with the Joneses' *American Economic Review Papers and Proceedings*, 80, 38–42.

Abel, A.B. and Bernanke, B.S. (1998) *Macroeconomics*, 3rd edition, Addison-Wesley: New York.

Aghion, P. and Howitt, P. (1994) 'Growth and Unemployment' *Review of Economic Studies*, 61, 477–94.

Alonso, W. (1964) *Location and Land Use*, Harvard University Press Cambridge: MA.

Alonso-Carrera, J., Caballé, J. and Raurich, X. (2005) 'Growth, Habit Formation, and Catching-up with the Joneses' *European Economic Review*, 49, 1665–91.

Anas, A. (1978) 'Dynamics of Urban Residential Growth', *Journal of Urban Economics*, 5, 66–87.

Anas, A. (1990) 'Taste Heterogeneity and Urban Spatial Structure: The Logit Model and Monocentric Theory Reconciled', *Journal of Urban Economics*, 28, 318–35.

Anas, A. (1992) 'On the Birth and Growth of Cities – Laissez-Faire and Planned Compared' *Regional Science and Urban Economics*, 22, 243–58.

Andersson, F. and Forslid, R. (2003) 'Tax Competition and Economic Geography', *Journal of Public Economic Theory*, 5, 279–303.

Arnott, R.J. (1980) 'A Simple Urban Growth Model with Durable Housing' *Regional Science and Urban Economics*, 10, 53–76.

Arnott, R.J. (1987) 'Economic Theory and Housing', *Handbook of Regional and Urban Economics*, Vol. II, edited by Mills, E.S., Amsterdam: North-Holland.

Arnott, R., Braid, R., Davidson, R. and Pines, D. (1999) 'A General Equilibrium Spatial Model of Housing Quality and Quantity', *Regional Science and Urban Economics*, 29, 283–316.

Arrow, K.J. (1962) 'The Economic Implications of Learning by Doing', *Review of Economic Studies*, 29, 155–73.

Asada, T. (1997) *Macroeconomics of Growth and Cycles*, Nihon Keizai Hyouron Sya (in Japanese): Tokyo.

Asada, T. (2007) 'Money in Economic Analysis', an article for the theme of Mathematical Economics in *Encyclopedia of Mathematical Sciences* (http://www.eolss.com), edited by the UNESCO, The UNESCO: Paris.

Asako, K. (1983) 'The Utility Function and the Superneutrality of Money on the Transition Path', *Econometrica*, 51, 1593–96.

Atkinson, A.B. and Bourguignon, F. (eds) (2000) *Handbook of Income Distribution*, Vol.1, Elsevier: Amsterdam.

Azam, J.P. (1983) 'Money, Growth and Disequilibrium', *Economica*, 50, 325–35.

Azariadis, C. (1993) *Intertemporal Macroeconomics*, Blackwell: Oxford.

Azariadis, C. and Lambertini, L. (2003) 'Endogenous Debt Constraints in Lifecycle Economies', *The Review of Economic Studies*, 70, 461–87.

Bain, K. and Howells, P. (2003) *Monetary Economics: Policy and Its Theoretical Basis*, Palgrave Macmillan: Houndmills.

Bairoch, P. (1993) *Economics and World History: Myths and Paradoxes*, University of Chicago Press: Chicago.

Bajo-Rubio, O. (2000) 'A Further Generalization of the Solow Growth Model: The Role of the Public Sector', *Economics Letters*, 68, 79–84.

Baldwin, R.E. and Martin, P. (2004) 'Agglomeration and Regional Growth', *Handbook of Regional and Urban Economics*, Vol. 4, edited by Henderson, J.V. and Thisse, J.F. Elsevier: Amsterdam.

Baldwin, R.E. and Krugman, P. (2004) 'Agglomeration, Integration and Tax Harmonization', *European Economic Review*, 48, 1–23.

Banerjee, A.V. and Newman, A.F. (1993) 'Occupational Choice and the Process of Development', *Journal of Political Economy*, 101, 274–98.

Barnett, R.C. (2005) 'Coordinating Macroeconomic Policy in a Simple AK Growth Model', *Journal of Macroeconomics*, 27, 621–47.

Barreto, R.A. (2000) 'Endogenous Corruption in a Neoclassical Growth Model', *European Economic Review*, 44, 35–60.

Barro, R. (1990) 'Government Spending in a Simple Model of Endogenous Growth', *Journal of Political Economy*, 98, S103–25.

Barro, R. (1991) 'Economic Growth in a Cross Section of Countries', *Quarterly Journal of Economics*, 106, 407–43.

Barro, R.J. (1995) 'Inflation and Growth', *Bank of England Quarterly Bulletin*, 1995, 39–52.

Barro, R.J. and Becker, G. (1989) 'Fertility Choice in a Model of Economic Growth', *Econometrica*, 57, 481–501.

Barro, R.J. and Sala-i-Martin, X. (1995) *Economic Growth*, McGraw-Hill, Inc.: New York.

Barten, A.P. and Böhm, V. (1982) 'Consumer Theory', *Handbook of Mathematical Economics II*, edited by Arrow, K.J. and Intriligator, M.D. North-Holland Publishing Company: Amsterdam.

Baumol, W. (1952) 'The Transactions Demand for Cash: An Inventory Theoretical Approach', *Quarterly Journal of Economics*, 67, 545–56.

Bayindir-Upmann, T. and Ziad, A. (2005) 'Existence of Equilibria in a Basic Tax Competition Model', *Regional Science and Urban Economics*, 35, 1–22.

Becker, G.S. (1965) 'A Theory of the Allocation of Time', *Economic Journal*, 75, 493–517.

Becker, G.S. (1981) *A Treatise on the Family*, Harvard University Press Mass: Cambridge.

Becker, R.A. (1992) 'Cooperative Capital Accumulation Games and the Core', *Economic Theory and International Trade: Essays in Memoriam, J. Trout Rader*, edited by Neuefeind, W. and Riezman, R.G., Springer-Verlag: Berlin.

Beckmann, M.J. (1969) 'On the Distribution of Urban Rent and Residential Density', *Journal of Economic Theory*, 1, 60–8.

Beckmann, M.J. (1976) 'Spatial Equilibrium in a Dispersed City', Mathematical Land Use Theory, edited by Papageorgiou, Y.Y., Lexington Books: Lexington.

Beckmann, M.J. and Papageorgiou, Y.Y. (1989) 'Heterogeneous Tastes and Residential Location' *Journal of Regional Science*, 29, 317–23.

Bénassy, J.P. (1975) 'Neokeynesian Disequilibrium in a Monetary Economy', *Review of Economic Studies*, 42, 503–24.

Bénassy, J.P. (1982) *The Economics of Market Disequilibrium*, Academic Press: New York.

Bénassy, J.P. (2007) 'IS-LM and the Multiplier: A Dynamic General Equilibrium Model', *Economics Letters*, 96, 189–95.

Benhabib, J. (2004) 'Interest Rate Policy in Continuous Time with Discrete Delays', *Journal of Money, Credit, and Banking*, 36, 1–15.

Benhabib, J. and Bull, C. (1983) 'The Optimal Quantity of Money: A Formal Treatment' *International Economic Review*, 24, 101–11.

Benhabib, J., Meng, Q. and Nishimura, K. (2000) 'Indeterminancy under Constant Returns to Scale in Multisector Economies', *Econometrica*, 68, 1541–8.

Benhabib, J. and Miyao, T. (1981) 'Some New Results of the Dynamics of the Generalized Tobin Model', *International Economic Review*, 22, 589–96.

Benhabib, J. and Perli, R. (1994) 'Uniqueness and Indeterminacy: On the Dynamics of Endogenous Growth', *Journal of Economic Theory*, 63, 113–42.

Benhabib, J., Rogerson, R. and Wright, R. (1991) 'Homework in Macroeconomics: Household Production and Aggregate Fluctuations', *Journal of Political Economy*, 99, 1166–87.

Benhabib, J., Schmitt-Grohé, S. and Uribe, M. (2001a) 'The Perils of Taylor Rules', *Journal of Economic Theory*, 96, 40–69.

Benhabib, J., Schmitt-Grohé, S. and Uribe, M. (2001b) 'Monetary Policy and Multiple Equilibria', *American Economic Review*, 91, 167–86.

Benhabib, J., Schmitt-Grohé, S. and Uribe, M. (2002a) 'Chaotic Interest-Rate Rules', *American Economic Review*, 92, 72–8.

Benhabib, J., Schmitt-Grohé, S. and Uribe, M. (2002b) 'Avoiding Liquidity Traps', *Journal of Political Economy*, 110, 535–63.

Benhabib, J. and Uribe, M. (2002) 'The Perils of Taylor's Rules', *Journal of Economic Theory*, 96, 40–69.

Berliant, M., Peng, S.K. and Wang, P. (2002) 'Production Externalities and Urban Configuration', *Journal of Economic Theory*, 104, 275–303.

Bewley, T. (1983) 'A Difficulty with the Optimum Quantity of Money', *Econometrica*, 51, 1485–504.

Bignami, F., Colombo, L. and Weinrich, G. (2004) 'Complex Business Cycles and Recurrent Unemployment in a Non-Walraisan Macroeconomic Model', *Journal of Economic Behavior & Organization*, 53, 173–91.

Blanchard, O. (1997) *Macroeconomics*, Prentice-Hall International: New Jersey.

Blanchard, O.J. and Fischer, S. (1989) *Lectures on Macroeconomics*, MIT Press: Cambridge, MA.

Boldrin, M. and Woodford, M. (1990) 'Equilibrium Models Displaying Endogenous Fluctuations and Chaos: A Survey', *Journal of Monetary Economics*, 25, 189–222.

Boldrin, M., Nishimura, Shigoka, T. and Yano. M. (2001) 'Chaotic Equilibrium Dynamics in Endogenous Growth Models', *Journal of Economics*, 96, 107–31.

Bond, E.W., Wang, P. and Yip, C.K. (1996) 'A General Two-Sector Model of Endogenous Growth with Human and Physical Capital: Balanced Growth and Transitional Dynamics', *Journal of Economic Theory*, 68, 149–73.

Boomsma, P. and Oosterhaven, J. (1992) 'A Double-Entry Method for the Construction of Bi-Regional Input-Output Tables', *Journal of Regional Science*, 32, 269–84.

Booth, A.L. (2002) *The Economics of Labor Unions*, Edward Elgar: Cheltenham.

Borck, R. and Pflüger, M. (2006) 'Agglomeration and Tax Competition', *European Economic Review*, 50, 647–68.

Borokov, E. and Hochman, O. (1977) 'Optimum and Market Equilibrium in a Model of a City Without a Predetermined Center', *Environment and Planning*, 9, 859–56.

Bouakez, H., Cardia, E. and Murcia, F.J.R. (2005) 'Habit Formation and the Persistence of Monetary Policy Shocks', *Journal of Monetary Economics*, 52, 1073–88.

Boyer, M. (1978) 'A Habit Forming Optimal Growth Model', *International Economic Review*, 19, 585–609.

Braid, R.M. (2001) 'Spatial Growth and Redevelopment with Perfect Foresight and Durable Housing', *Journal of Urban Economics*, 49, 425–52.

Brecher, R.A., Chen, Z.Q. and Choudhri, E.U. (2002) 'Absolute and Comparative Advantage, Reconsidered: The Pattern of International Trade with Optimal Saving', *Review of International Economics*, 10, 645–56.

Bresnahan, T.F. and Trajtenberg, M. (1995) 'General Purpose Technologies: 'Engines of Growth'?, *Journal of Econometrics*, 65, 83–108.

Brezis, E.S., Krugman, P.R. and Tsiddon, D. (1993) 'Leapfrogging in International Competition: A Theory of Cycles in National Technological Leadership', *American Economic Review*, 83, 1211–19.

Brito, P.M.B. and Pereira, A.M. (2002) 'Housing and Endogenous Long-Term Growth', *Journal of Urban Economics*, 51, 246–71.

Brock, W.A. (1974) 'Money and Growth: The Case of Long Run Perfect Foresight',*International Economic Review*, 15, 750–77.

Brock, W.A. and Turnovsky, S.J. (1981) 'The Analysis of Macroeconomic Policies in Perfect Foresight Equilibrium', *International Economic Review*, 22, 179–209.

Brueckner, J.K. (1981) 'A Dynamic Model of Housing Production', *Journal of Urban Economics*, 10, 1–14.

Brueckner, J.K. and Pereira, A.M. (1994) 'Housing Ownership and the Business Cycle', *Journal of Housing Economics*, 3, 165–85.

Buiter, W.H. and Miller, M.H. (1982) 'Real Exchange Rate Overshooting and the Output Cost of Bringing Down Inflation', *European Economic Review*, 18, 85–123.

Bunzel, H. (2006) 'Habit Persistence, Money, and Overlapping Generations', *Journal of Economic Dynamics & Control*, 30, 2425–45.

Burbidge, J. and Cuff, K. (2005) Capital Tax Competition and Returns to Scale', *Regional Science and Urban Economics*, 35, 353–73.

Burmeister, E. and Dobell, A.R. (1970) *Mathematical Theories of Economic Growth*, Collier Macmillan Publishers: London.

Cagan, P. (1956) 'The Monetary Dynamics of Hyperinflation', *Studies in the Quantity Theory of Money*, edited by Friedman, M., University of Chicago Press: Chicago.

Calvo, G.A. (1979) 'On Models of Money and Perfect Foresight', *International Economic Review*, 20, 83–103.

Campbell, J.Y. and Ludvigson, S. (2001) 'Elasticities of Substitution in Real Business Cycle Model with Home Production', *Journal of Money, Credit, and Banking*, 33, 847–75.

Canning, P. and Wang, Z. (2005) 'A Flexible Mathematical Programming Model to Estimate Interregional Input-Output Accounts', *Journal of Regional Science*, 45(3), 539–63.

Capello, R. and Nijkamp, P. (eds) (2004) *Urban Dynamics and Growth: Advances in Urban Economics*, Elsevier: Amsterdam.

Caplin, A.S. and Spulber, D.F. (1987) 'Menu Costs and the Neutrality of Money' *Quarterly Journal of Economics*, 106, 683–708.

Carlstrom, C.T. and Fuerst, T.S. (2005) 'Investment and Interest Rate Policy: A Discrete-Time Analysis', *Journal of Economic Theory*, 123, 4–20.

Carroll, C., Overland, J. and Weil, D. (1997) 'Comparison Utility in a Growth Model', *Journal of Economic Growth*, 2, 229–67.

Carroll, C., Overland, J. and Weil, D. (2000) 'Saving and Growth with Habit Formation', *American Economic Review*, 84, 1397–408.

Cass, D. (1965) 'Optimum Growth in an Aggregative Model of Capital Accumulation', *Review of Economic Studies*, 32, 233–40.

Cass, D. and Shell, K. (1976) 'The Structure and Stability of Competitive Dynamical Systems', *Journal of Economic Theory*, 12, 31–70.

Cass, D. and Shell, K. (1980) 'In Defense of a Basic Approach', *Model of Monetary Economics*, edited by J.H. Kareken and N. Wallace, Federal Reserve Bank of Minneapolis: Minneapolis.

Chakraborty, S. (2004) 'Endogenous Lifetime and Economic Growth', *Journal of Economic Theory*, 116, 119–37.

Champ, B. and Freeman, S. (2001) *Modeling Monetary Economies*, Cambridge University Press: Cambridge.

Chang, W.Y. (2002) 'Examining the Long-Run Effect of Money on Economic Growth: An Alternative View', *Journal of Macroeconomics*, 24, 81–102.

Chang, W.Y. and Lai, C.C. (2000) 'Anticipated Inflation in a Monetary Economy with Endogenous Growth', *Economica*, 67, 399–417.

Chang, W.Y., Tsai, H.F. and Liu, W.F. (1998a) 'Effects of Government Spending on the Current Account with Endogenous Time Preference', *Southern Economic Journal*, 64, 728–40.

Chang, W.Y., Tsai, H.F. and Liu, W.F. (1998b) 'Government Spending and Capital Accumulation with Endogenous Time Preference', *The Canadian Journal of Economics*, 31, 624–45.

Chari, V.V. and Hopenhayn, H. (1991) 'Vintage Human Capital, Growth, and the Diffusion of New Technology', *Journal of Political Economy*, 99, 1142–65.

Chatterjee, S., Sakoulis, G. and Turnovsky, S.J. (2003) 'Unilateral Capital Transfers, Public Investment, and Economic Growth', *European Economic Review*, 47, 1077–103.

Chiang, A.C. (1984) *Fundamental Methods of Mathematical Economics*, McGraw-Hill Book Company: London.

Chiappori, P.A., Fortin, B. and Lacroix, G. (2002) 'Marriage Market, Divorce Legislation, and Household Labor Supply', *Journal of Political Economy*, 110, 37–72.

Chiarella, C. and Flaschel, P. (2000) *The Dynamics of Keynesian Monetary Growth: Macro Foundations*, Cambridge University Press: Cambridge.

Chirichiello, G. (2000) *Intertemporal Macroeconomic Models, Money and Rational Choices*, Macmillan: London.

Christiano, L.J., Eichenbaum, M.S. and Evans, C. (2005) 'Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy', *Journal of Political Economy*, 113, 1–45.

Clarida, R., Gali, J. and Gertler, M. (1998) 'Monetary Policy Rules in Practice: Some International Evidence', *European Economic Review*, 42, 1033–67.

Clower, R.W. (1965) 'The Keynesian Counterrevolution – A Theoretical Appraisal', *The Theory of Interest Rates*, edited by Hahn, F.H. and Brechling, F.P.R., Macmillan: London.

Clower, R.W. (1967) 'A Reconsideration of the Microfoundations of Monetary Theory', *Western Economic Journal*, 6, 1–9.

Cole, H.L. and Stockman, A.C. (1992) 'Specification, Transactions Technologies, and Money Growth', *International Economic Review*, 33, 283–98.

Combes, P.G., Duraton, G. and Gobillon, L. (2003) 'Spatial Wage Disparities: Sorting Matters!', *CEPR Discussion Paper*, 4220.

Constantinides, G.M. (1990) 'Habit Formation: A Resolution of the Equity Premium Puzzle', *Journal of Political Economy*, 98, 519–43.

Coppel, W.A. (1955) 'The Solution of Equations by Iteration', *Proceedings of the Cambridge Philosophical Society*, 51, 41–3.

Corden, W.M. (1966) 'The Two Sector Growth Model with Fixed Coefficients', *Review of Economic Studies*, 33, 253–63.

Croushore, D. (1993) 'Money in the Utility Function: Functional Equivalence to a Shopping-Time Model', *Journal of Macroeconomics*, 15, 175–82.

Das, M. (2003) 'Optimal Growth with Decreasing Marginal Impatience', *Journal of Economic Dynamics* & Control, 27, 1881–91.

Davis, L.S. (2003) 'The Division of Labor and the Growth of Government', *Journal of Economic Dynamics & Control*, 27, 1217–35.

Davis, S.J. and Haltiwanger, J. (1992) 'Gross Job Creation, Cross Job Destruction, and Employment Reallocation', *Quarterly Journal of Economics*, 107, 819–64.

Day, R.H. (1984) *Complex Economic Dynamics*, The MIT Press: Cambridge, MA.

Debreu, G. (1959) *Theory of Value*, John Wiley & Sons, Inc.: New York.

De la Croix, D. and Michel, P. (2002) *A Theory of Economic Growth: Dynamics and Policy in Overlapping Generations*, Cambridge University Press: Cambridge.

Dessing, M. (2002) 'Labor Supply, the Family and Poverty: The S-shaped Labor Supply Curve', *Journal of Economic Behavior & Organization*, 49, 433–58.

Dessing, M. (2004) 'Implications for Minimum-Wage Policies of an S-shaped Labor-Supply Curve', *Journal of Economic Behavior & Organization*, 53, 543–68.

Devereux, M.B., Head, A.C., and Lapham, B.J. (1996) 'Monopolistic Competition, Increasing Returns, and the Effects of Government Spending', *Journal of Money, Credit, and Banking*, 28, 233–54.

Devereux, M.B., Head, A.C., and Lapham, B.J. (2000) 'Government Spending and Welfare with Returns to Specification', *Scandinavian Journal of Economics*, 102, 547–61.

Devereux, M.B., Shi, K. and Xu J.Y. (2006) 'Global Monetary Policy under a Dollar Standard', *Journal of International Economics*, 71, 113–32.

Diamond, P.A. (1965a) 'National Debt in a Neoclassical Growth Model', *American Economic Review*, 55, 1125–50.

Diamond, P.A. (1965b) 'Disembodied Technical Change in a Two-Sector Model', *Review of Economic Studies*, 32, 161–8.

Díaz, A., Pijoan-Mass, J. and Rios-Rull, J.V. (2003) 'Precautionary Savings and Wealth Distribution under Habit Formation Preferences', *Journal of Monetary Economics*, 50, 1257–91.

DiPasquale, D. and Wheaton, W.C. (1996) *Urban Economics and Real Estate Markets*, Prentice Hall: Englewood Cliffs.

Dixit, A. (2003) 'On Modes of Economic Governance', *Econometrica*, 71, 449–81.

Dollar, D. (1986) 'Technological Innovation, Capital Mobility, and the Product Cycle in the North-South Trade', *American Economic Review*, 76, 177–90.

Dornbusch, R. (1976) 'Expectations and Exchange Rate Dynamics', *Journal of Political Economy*, 84, 1161–76.

Dornbusch, R. and Fischer, S. (1973) 'Inflation and Growth' *Journal of Money, Credit and Banking*, 5, 141–56.

Dornbusch, R. and Fischer, S. (1980) 'Exchange Rate and the Current Account', *American Economic Review*, 70, 960–71.

Dornbusch, R. and Frenkel, J.A. (1973) 'Inflation and Growth: Alternative Approaches', *Journal of Money, Credit and Banking*, 5, 141–56.

Dornbusch, R. and Giovannini, A. (2000) 'Monetary Policy in the Open Economy', *Handbook of Monetary Economics*, vol. II, edited by Friedman, B.M. and Hahn, F. Elsevier: Amsterdam.

Dowd, K. (1990) 'The Value of Time and the Transactions Demand for Money', *Journal of Money, Credit and Banking*, 22, 51–64.

Drandakis, E. (1963) 'Factor Substitution in the Two-Sector Growth Model', *Review of Economic Studies*, 30, 105–18.

Dréze, J. (1975) 'Existence of an Exchange Equilibrium under Price Rigidities', *International Economic Review*, 42, 301–20.

Drugeon, J.P. and Venditti, A. (2001) 'Intersectoral External Effects, Multiplicies & Indeterminacies', *Journal of Economic Dynamics and Control*, 25, 765–87.

Duranton, G. and Monastiriotis, V. (2002) 'Mind the Gaps: The Evolution of Regional Earnings Inequalities in the UK 1982–1997', *Journal of Regional Science*, 42, 219–56.

Dutkowsky, D.H. and Dunsky, R.M. (1996) 'Intertemporal Substitution, Money, and Aggregate Labor Supply', *Journal of Money, Credit and Banking*, 28, 216–32.

Dutkowsky, D.H. and Foote, W.G. (1996) 'Intertemporal Substitution in Macroeconomics: Consumption, Labor Supply, and Money Demand', *Review of Economics and Statistics*, 74, 333–8.

Easterly, W. and Fischer, S. (2001) 'Inflation and Poor', *Journal of Money, Credit and Banking*, 33, 160–78.

Eden, F. (2005) *A Course in Monetary Economics: Sequential Trade, Money, and Uncertainty*, Blackwell Publishing: Malden.

Ehrilich, I. and Lui, F.T. (1991) 'Intergenerational Trade, Longevity and Economic Growth', *Journal of Political Economy*, 99, 1029–59.

Erceg, C. and Levin, A. (2006) 'Optimal Monetary Policy with Durable Consumption Goods', *Journal of Monetary Economics*, 53, 1341–59.

Espinosa-Vega, M.A. and Yip, C.Y. (1999) 'Fiscal and Monetary Policy Interactions in an Endogenous Growth Model with Financial Intermediaries', *International Economic Review*, 40, 595–615.

Faria, J.R. (2001) 'Habit Formation in a Monetary Growth Model', *Economics Letters*, 73, 51–5.

Faria, J.R. (2002) 'Growth and Labor Supply in the Presence of Habit Formation in Consumption', *Economics Bulletin*, 4, 1–5.

Farmer, R.E.A. (1986) 'Deficits and Cycles', *Journal of Economic Theory*, 40, 77–88.

Feenstra, R.C. (1976) 'Inflation, Income Taxes, and the Rate of Interest: A Theoretical Analysis', *American Economic Review*, 66, 809–20.

Feenstra, R.C. (1980) 'Inflation, Tax Rules, and the Stock Market', *Journal of Monetary Economics*, 6, 309–31.

Feenstra, R.C. (1986) 'Functional Equivalence between Liquidity Costs and the Utility of Money', *Journal of Monetary Economics*, 17, 271–91.

Fischer, S. (1974) 'Money and the Production Function', *Economic Inquiry*, 12, 517–33.

Fischer, S. (1979a) 'Capital Accumulation on the Transition Path in a Monetary Optimizing Model', *Econometrica*, 47, 1433–9.

Fischer, S. (1979b) 'Anticipated and the Nonneutrality of Money', *Journal of Political Economy*, 87, 225–52.

Fleming, J.M. (1962) 'Domestic Fiscal Policy under Fixed and Under Floating Exchange Rate', *IMF Staff Papers*, 9, 369–79.

Forslid, R. and Ottaviano, G.I.P. (2003) 'An Analytically Solvable Core-Periphery Model', *Journal of Economic Geography*, 3, 229–40.

Franke, R. and Asada, T. (1994) 'A Keynes-Goodwin Model of the Business Cycle', *Journal of Economic Behavior and Organization*, 24, 273–95.

Frederick, S., Loewenstein, G. and O'Donoghue, T. (2002) 'Time Discontinuing and Time Preference: A Critical Review', *Journal of Economic Literature*, 40, 351–401.

Frenkel, J.A. (1979) 'On the Mark: A Theory of Floating Exchange Rates Based on Real Interest Differentials', *American Economic Review*, 69, 610–22.

Frenkel, J. and Razin, A. (1987) *Fiscal Policy and the World Economy*, MIT Press: Cambridge, MA.

Frenkel, J.A., Razin, A. and Sadka, E. (1991) 'International Taxation in an Integrated World', MIT Press: Cambridge, MA.

Frenkel, J.A., Razin, A. and Yuen, C.W. (1996) '*Fiscal Policies and Growth in the World Economy*', MIT Press: Cambridge, MA.

Friedman, M. (1956) 'The Quantity Theory of Money: A Restatement', *Studies in the Quantity Theory of Money*, edited by Friedman M., University of Chicago Press: Chicago.

Friedman, M. (1964) 'Post-War Trends in Monetary Theory and Practice, National Banking Review 2', reprinted in *The Optimum Quantity of Money and Other Essays*, Aldine: Chicago.

Friedman, M. (1969) 'The Optimum Quantity of Money', in *The Optimum Quality of Money and Other Essays*, edited by Friedman, M., Aldine: Chicago.

Friedman, M. (1970) 'A Theoretical Framework of Political Economy', *Journal of Political Economy*, 78, 193–238.

Froyen, R.T. (1999) Macroeconomics: Theories and Policies, 6th edition. Prentice Hall International Inc: Singapore.

Fuhrer, J.C. (2000) 'Habit Formation in Consumption and Its Implications for Monetary-Policy Models', *American Economic Review*, 90, 367–90.

Fujita, M. (1989) *Urban Economic Theory – Land Use and City Size*, Cambridge University Press: Cambridge.

Fujita, M., Krugman, P. and Venables, A.J. (1999) *The Spatial Economy – Cities, Regions, and International Trade*, The MIT Press: Cambridge, MA.

Fujita, M. and Thisse, J.F. (2002a) 'Does Geographical Agglomeration Foster Economic Growth? And Who Gains and Loses from It?', *Japanese Economic Review*, 54, 121–45.

Fujita, M. and Thisse, J.F. (2002b) *Economics of Agglomeration: Cities, Industrial Location, and Regional Growth*, Cambridge University Press: Cambridge.

Gale, D. (1983) *Money: In Disequilibrium*, Cambridge University Press: Cambridge.

Galor, O. and Ryder, H.E. (1989) 'Existence, Uniqueness, and Stability of Equilibrium in an Overlapping-Generations Model with Productive Capital' *Journal of Economic Theory*, 49, 360–75.

Gandolfo, G. (2001) *International Finance and Open-Economy Macroeconomics*, Springer: Berlin.

Gandolfo, G. (2007) 'Models of International Economics, an article for the theme of Mathematical Models in Economics', *Encyclopedia of Mathematical Sciences* (http://www.eolss.com), edited by the UNESCO, The UNESCO: Paris.

Ganelli, G. (2007) 'Fiscal Policy Rules in an Overlapping Generations Model with Endogenous Labour Supply', *Journal of Economic Dynamics & Control*, 31, 1015–35.

Glaeser, E.L. and Gyourko, J. (2005) 'Urban Decline and Durable Housing', *Journal of Political Economy*, 113, 345–75.

Glaeser, E.L., Kolko, J. and Saiz, A. (2001) 'Consumer City', *Journal of Economic Geography*, 1, 27–50.

Glaeser, E.L. and Mare, D.C. (2001) 'Cities and Skills', *Journal of Labor Economics*, 19, 316–42.

Glomm, G. and Ravikumar, B. (1994) 'Public Investment in Infrastructure in a Simple Growth Model', *Journal of Economic Dynamics & Control*, 18, 1173–87.

Glomm, G. and Ravikumar, B. (1997) 'Productive Government Expenditures and Long-Run Growth', *Journal of Economic Dynamics & Control*, 21, 183–204.

Gómez, M.A. (2006) 'Optimal Consumption Taxation in a Model of Endogenous Growth with External Habit Formation', *Economics Letters* , 93, 427–35.

Gomme, P. (1993) 'Money and Growth Revisted: Measuring the Costs of Inflation in an Endogenous Growth Model', *Journal of Monetary Economics*, 32, 51–77.

Gram, H.N. (1967) 'Two-Sector Models in the Theory of Capital and Growth', *American Economic Review*, 66, 891–903.

Grandmont, J.M. (1983) *Money and Value*, Cambridge University Press: Cambridge.

Grandmont, J.M. (1985) 'On Endogenous Competitive Business Cycles', 53, 535–72.

Grandmont, J.M. (1998) 'Expectations Formation and Stability of Large Socio-economic Systems', *Econometrica*, 66, 741–81.

Grandmont, J.M. and Laroque, G. (1991) 'Economic Dynamics with Learning: Some Instability Examples', in *Equilibrium Theory and Applications*, edited by Barnett, W.A. *et al.*, Cambridge University Press: Cambridge.

Gray, J.S. (1998) 'Divorce-Law Changes, Household Bargaining, and Married Women's Labor Supply', *American Economic Review*, 88, 628–42.

Gray, M.R. and Turnovsky, S.J. (1979) 'The Stability of Exchange Rate Dynamics under Myopic Perfect Foresight', *International Economic Review*, 16, 199–221.

Green, J.R. and Laffont, J.J. (1981) 'Disequilibrium Dynamics with Inventories and Anticipatory Price-Setting', *European Economic Review*, 16, 199–211.

Greenwood, J. and Hercowitz, Z. (1991) 'The Allocation of Capital and Time over the Business Cycle', *Journal of Political Economy*, 99, 1188–214.

Guo, J.T. (2004) 'Increasing Returns, Capital Utilization, and the Effects of Government Spending', *Journal of Economic Dynamics & Control*, 28, 1059–78.

Hadjimichalakis, M. (1971a) 'Money, Expectations, and Dynamics – An Alternative View', *International Economic Review*, 12, 381–402.

Hadjimichalakis, M. (1971b) 'Equilibrium and Disequilibrium Growth with Money – The Tobin Models', *Review of Economic Studies*, 38, 457–79.

Hafer, R.W., Haslag, J.H. and Jones, G. (2007) 'On Money and Output: Is Money Redundant?', *Journal of Monetary Economics*, 54, 945–54.

Hahn, F.H. (1965) 'On Some Problems of Proving the Existence of Equilibrium in a Monetary Economy', *The Theory of Interest Rates*, edited by Hahn, F.H. and Brechling, F.P.R., Macmillan: London.

Hahn, F.H. (1969) 'On Money and Growth', *Journal of Money. Credit and Banking*, 1, 172–87.

Hahn, F.H. (1978) 'On Non-Walrasian Equilibria', *Review of Economic Studies*, 45, 1–18.

Hahn, F.H. (2000) 'Liquidity', in *Handbook of Monetary Economics*, Vol. I, edited by Friedman, B.M. and Hahn, F., Elsevier: Amsterdam.

Hahn, F.H. and Solow, R.M. (1985) *A Critical Essay on Modern Macroeconomic Theory*, Blackwell: Oxford.

Hale, J. and Koçak, H. (1991) *Dynamics and Bifurcations*, Springer-Verlag: Berlin.
Handa, J. (2005) *Monetary Economics*, Routledge: Oxon.
Hanson, G.H. (2001) 'Scale Economies and the Geographic Concentration of Industry', *Journal of Economic Geography*, 1, 255–76.
Heijdra, B. and Ligthart, J. (2000) 'The Dynamic Macroeconomic Effects of Tax Policy in an Overlapping Generations Model', *Oxford Economic Papers*, 52, 677–701.
Henderson, J.V. (1974) 'The Sizes and Types of Cities', *American Economic Review*, 64, 640–56.
Henderson, J.V., Schalizi, Z. and Venables, A.J. (2001) 'Geography and Development', *Journal of Economic Geography*, 1, 81–105.
Henderson, J.V. and Thisse, J.F. (eds) (2004) *Handbook of Regional and Urban Economics*, Elsevier: Amsterdam.
Henderson, J.V. (1985) *Economic Theories and Cities*, Academic Press: New York.
Hicks, J. (1937) 'Mr. Keynes and the Classics: A Suggested Interpretation', *Econometrica*, 5, 137–59.
Hirschman, A.O. (1958) *The Strategy of Development*, Yale University Press; New Haven.
Ho, W.M., Zeng, J.L. and Zhang, J. (2007) 'Inflation Taxation and Welfare with Externalities and Leisure', *Journal of Money, Credit, and Banking*, 39, 105–31.
Hockman, O. and Pines, D. (1980a) 'Costs Adjustment and Demolition Costs in Residential Construction and Their Effects on Urban Growth', *Journal of Urban Economics*, 7, 2–19.
Hickman, O. and Pines, D. (1980b) 'Costs of Adjustment and the Spatial Pattern of a Growing Open City', *Econometrica*, 50, 1371–89.
Hommes, C.H. (1991) *Chaotic Dynamics in Economic Models: Some Simple Case-Studies*, Wolters-Noordhoff: Groningen.
Hommes, C.H. (1998) 'On the Consistency of Backward-Looking Expectations: The Case of the Cobweb', *Journal of Economic Behavior & Organization*, 33, 333–62.
Huang, K.X.D. and Meng, Q.L. (2007) 'Capital and Macroeconomic Instability in a Discrete-time Model with Forward-Looking Interest Rate Rules', *Journal of Economics & Control*, 31, 2802–26.
Hume, D. (1970) *Writings on Economics*, edited by Rotwein E., University of Wisconsin Press: Madison.
Huo, T.M. (1977) 'Inflation and Capital Accumulation in a Two-Sector Cash-in-Advance Economy', *Journal of Macroeconomics*, 19, 103–15.
Ianchovichina, E. and McDougall, R. (2001) 'Theoretical Structure of Dynamic GTAP', *GTAP Technical Paper*, No. 17.
Ikeda, S. and Ono, Y. (1992) 'Macroeconomic Dynamics in a Multi-Country Economy – A Dynamic Optimization Approach', *International Economic Review*, 33, 629–44.
Imai, H. (1982) 'CBD Hypothesis and Economies of Agglomeration', *Journal of Economic Theory*, 28, 275–99.
Imrohproğlu, A. (1992) 'The Welfare Costs of Inflation Under Imperfect Insurance', *Journal of Economic Dynamics and Control*, 16, 79–91.
Iooss, G. and Joseph, D.D. (1980) *Elementary Stability and Bifurcation Theory*, Springer-Verlag: New York.
Ireland, P.N. (1994a) 'Supply-side Economics and Endogenous Growth', *Journal of Monetary Economics*, 33, 557–71.
Ireland, P. N. (1994b) 'Money and Growth: An Alternative Approach', *American Economic Review*, 84, 47–65.

Isard, W. (1953) 'Some Empirical Results and Problems of Regional Input-Output Analysis', Leontief, W. *et al.*, *Studies in the Structure of the American Economy*, Oxford University Press: New York.

Jellah, M. and Zenou, Y. (2000) 'A Dynamic Efficiency Wage Model with Learning by Doing', *Economic Letters*, 66, 99–105.

Jha, R. (2003) *Macroeconomics for Developing Countries*, Routledge: New York.

Jha, S.K., Wang, P. and Yip, C.K. (2002) 'Dynamics in a Transactions-based Monetary Growth Model', *Journal of Economic Dynamics & Control*, 26, 611–35.

Johnson, H.G. (1969) 'Inside Money, Outside Money, Income, Wealth and Welfare in Monetary Theory', *Journal of Money, Credit and Banking*, 1, 30–45.

Jones, L. and Manuelli, R. (1990) 'A Convex Model of Equilibrium Growth: Theory and Policy Implications', *Journal of Political Economy*, 98, 1008–38.

Jovanovic, B. and Nyarko, Y. (1996) 'Learning by Doing and the Choice of Technology' *Econometrica*, 64, 1299–310.

Jullien, B. (1988) 'Competitive Business Cycles in an Overlapping Generations Economy with Productive Investment', *Journal of Economic Theory*, 46, 45–65.

Kam, E. and Missios, P. (2003) 'Wealth Effects in a Cash-in-Advance Economy', *Economics Bulletin*, 5, 1–7.

Kanemoto, Y. (1980) *Theories of Urban Externalities*, North-Holland: Amsterdam.

Karp, L. and Lee, H. (2001) 'Learning by Doing and the Choice of Technology: The Role of Patience', *Journal of Economic Theory*, 100, 73–92.

Kennedy, C. (1964) 'Induced Bias in Innovation and the Theory of Distribution', *Economic Journal*, 74, 541–7.

Keynes, J.M. (1936) *The General Theory of Employment, Interest Rate and Money*, Harcourt, Brace: New York.

Kimbrough, K.P. (1986) 'Inflation, Employment, and Welfare in the Presence of Transaction Costs', *Journal of Money, Credit, and Banking*, 18, 127–39.

King, R.G. and Rebelo, S. (1990) 'Public Policy and Economic Growth: Developing Neoclassical Implications', *Journal of Political Economy*, 98, S126–50.

Kiyotaki, N. and Wright, R. (1989) 'On Money as a Medium of Exchange', *Journal of Political Economy*, 97, 927–54.

Koopmans, T.C. (1965) 'On the Concept of Optimal Economic Growth', *The Econometric Approach to Development Planning*, North Holland: Amsterdam.

Kouri, P.J.K. (1974) 'The Exchange Rate and the Balance of Payments in the Short Run and in the Long Run: A Monetary Approach', *Scandinavian Journal of Economics*, 78, 280–304.

Krugman, P. (1979) 'A Model of Balance-Payments Crisis', *Journal of Money, Credit and Banking*, 11, 311–25.

Krugman, P. (1991) 'Increasing Returns and Economic Geography', *Journal of Political Economy*, 99, 483–99.

Krugman, P., Persson, T. and Svensson, L.E. (1985) 'Inflation, Interest Rates, and Welfare', *Quarterly Journal of Economics*, 100, 677–95.

Krusell, P. and Rios-Rull, J.V. (1996) 'Vested Interests in a Positive Theory of Stagnation and Growth', *Review of Economic Studies*, 63, 301–29.

Krusell, P. and Rios-Rull, J.V. (1999) 'On the Size of U.S. Government: Political Economy in the Neoclassical Growth Model', *American Economic Review*, 89, 1156–81.

Kuhn, T.S. (1977) *The Essential Tension*, University of Chicago Press: Chicago.

Kuznets, S. (1966) *Modern Economic Growth – Rate, Structure, and Spread*, Yale University Press: London.

Kuznetsov, Y.A. (1998) *Elements of Applied Bifurcation Theory*. Springer: Berlin.

Ladrón-de-Guevara, A., Ortigueira, S. and Santos, M.S. (1997) 'Equilibrium Dynamics in Two-Sector Models of Endogenous Growth', *Journal of Economic Dynamics and Control*, 21, 115–43.

Lahiri, A. and Puhakka, M. (1998) 'Habit Persistence in Overlapping Generations Economies Under Pure Exchange', *Journal of Economic Theory*, 78, 178–86.

Lancaster, K. (1966) 'A New Approach to Consumer Theory', *Journal of Political Economy*, 74, 132–57.

Lancaster, K. (1971) *Consumer Demand*, Columbia University Press: New York.

Leeper, E. (1991) 'Equilibria under "Active" and "Passive" Monetary and Fiscal Policies', *Journal of Monetary Economics*, 27, 129–47.

Leibenstein, H. (1957) *Economic Backwardness and Economic Growth: Studies in the Theory of Economic Development*, Wiley: New York.

Leijonhufvud, A. (1968) *On Keynesian Economics and the Economics of Keynes*, Oxford University Press: London.

Leith, C. and Wren-Lewis, S. (2000) 'Interaction between Monetary and Fiscal Policy Rules', *Economic Journal*, 110, 93–108.

Levhari, D. and Patinkin, D. (1968) 'The Role of Money in a Simple Growth Model', *American Economic Review*, 58, 713–53.

Lin, C.C., Mai, C.C. and Wang, P. (2004) 'Urban Land Policy and Housing in an Endogenously Growing Monocentric City', *Regional Science and Urban Economics*, 34, 241–61.

Lioui, A. and Rangvid, J. (2007) 'Habit Persistence in Consumption and the Demand for Money', *Economics Letters*, 96, 168–76.

Liso, N.D., Filatrella, G. and Weaver, N. (2001) 'On Endogenous Growth and Increasing Returns: Modeling Learning-by-Doing and the Division of Labor', *Journal of Economic Behavior & Organization*, 46, 39–55.

Love, D.R.F. and Wen, J.F. (1999) 'Inflation, Welfare, and Time-Costs of Transacting', *Canadian Journal of Economics*, 32, 171–94.

Lucas, R.E. (1972) 'Expectations and the Neutrality of Money', *Journal of Economic Theory*, 4, 102–24.

Lucas, R.E. (1973) 'Some International Evidence on Output-Inflation Tradeoffs', *American Economic Review*, 63, 326–34.

Lucas, R.E. (1988) 'On the Mechanics of Economic Development', *Journal of Monetary Economics*, 22, 3–42.

Lucas, R.E. Jr. (1996) 'Nobel Lecture: Monetary Neutrality', *Journal of Political Economy*, 104, 661–82.

Lucas, R.E. and Rossi-Hansberg, E. (2002) 'On the Internal Structure of Cities', *Econometrica*, 70, 1445–76.

MaCandless, G.T. and Weber, W.E. (1995) 'Some Monetary Facts', *Federal Reserve Bank of Minneapolis Quarterly Review*, 19, 2–11.

Makris, M. (2006) 'Capital Tax Competition under a Common Currency', *Journal of Urban Economics*, 59, 54–74.

Malinvaud, E. (1977) *The Theory of Unemployment Reconsidered*, Blackwell: Oxford.

Malinvaud, E. (1998) *Macroeconomic Theory – Economic Growth and Short-Term Equilibrium*, Elsevier: Singapore.

Mansoorian, A. (1996) 'On the Macroeconomic Policy Implications of Habit Persistence', *Journal of Money, Credit and Banking*, 28, 119–29.

Mansoorian, A. (1998) 'Habits and Durability in Consumption, and the Dynamics of the Current Account', *Journal of International Economics*, 44, 69–82.

Marimon, R., Nicolini, J.P. and Teles, P. (2003) 'Inside-Outside Money Competition', *Journal of Monetary Economics*, 50, 1701–18.

Marini, G. and van der Ploeg, F. (1988) 'Monetary and Fiscal Policy in an Optimising Model with Capital Accumulation and Finite Lives', *The Economic Journal*, 98, 772–86.

Marquist, M.H. and Reffett, K.L. (1991) 'Real Interest Rates and Endogenous Growth in a Monetary Economy', *Economics Letters*, 37, 105–9.

Marquist, M.H. and Reffett, K.L. (1995) 'The Inflation Tax in a Convex Model of Equilibrium Growth', *Econometrica*, 62, 109–21.

Marty, A.L. (1969) 'Inside Money, Outside Money, and Wealth Effect: A Review Essay', *Journal of Money, Credit and Banking*, 1, 101–11.

Mas-Colell, A., Whinston, M.D. and Green, J.R. (1995) *Microeconomic Theory*, Oxford University Press: New York.

Matsuyama, K. (1991) 'Endogenous Price Fluctuations in an Optimizing Model of a Monetary Economy', *Econometrica*, 59, 1617–31.

Matsuyama, K. (2001) 'Growing Through Cycles in an Infinitely Lived Agent Economy', *Journal of Economic Theory*, 100, 220–34.

Mazumdar, D. (1959) 'The Marginal Productivity Theory of Wages and Disguised Unemployment', *Review of Economic Studies*, 26, 190–7.

McCallum, B.T. (1983) *Monetary Economics, Theory and Policy*, Macmillan: New York.

McCann, P. (2005) 'Transport Costs and New Economic Geography', *Journal of Economic Geography*, 5, 305–18.

McDougall, R. (2002) 'A New Regional Household Demand System for GTAP', *GTAP Technical Paper*, No. 20.

Meng, Q.L. (2002) 'Monetary Policy and Multiple Equilibria in a Cash-in-Advance Economy', *Economics Letters*, 74, 165–70.

Miller, M.H. and Orr, D. (1966) 'A Model of the Demand for Money by Firms', *Quarterly Journal of Economics*, 80, 413–35.

Mills, E.S. (1967) 'An Aggregative Model of Resource Allocation in a Metropolitan Area', *American Economic Review*, 57, 197–210.

Mino, K. (1996) 'Analysis of a Two-Sector Model of Endogenous Growth with Capital Income Taxation', *International Economic Review*, 37, 227–51.

Mino, K. and Shibata, A. (1995) 'Monetary Policy, Overlapping Generations, and Patterns of Growth', *Economica*, 62, 179–94.

Modigliani, F. (1966) 'The Life Cycle Hypothesis of Saving, the Demand for Wealth and the Supply of Capital', *Social Research*, 33, 160–217.

Mohsin, M. (2006) 'Durability in Consumption and the Dynamics of the Current Account', *Journal of Economic Dynamics & Control*, 30, 143–62.

Mundell, R.A. (1960) 'The Monetary Dynamics of International Adjustment under Fixed and Flexible Exchange Rates', *Quarterly Journal of Economics*, 74, 227–57.

Mundell, R.A. (1965) 'Growth, Stability, and Inflationary Finance', *Journal of Political Economy*, 73, 97–109.

Mussa, M. (1981) 'Sticky Prices and Disequilibrium Adjustment in a Rational Model of the Inflationary Process', *American Economic Review*, 71, 1020–27.

Muth, R.F. (1969) *Cities and Housing*, University of Chicago Press: Chicago.

Muth, R.F. (1973) 'A Vintage Model of the Housing Stock', *Papers of the Regional Science Association*, 30, 141–56.

Myrdal, G. (1957) *Rich Lands and Poor*, Harper and Rowe: New York.

Nardini, F. (2001) *Technical Progress and Economic Growth: Business Cycles and Stabilization Policies*, Springer: Berlin.

Negishi, T. (1985) *Economic Theories in a Non-Walrasian Tradition*, Cambridge University Press: Cambridge.

Nelson, R.R. and Winter, S.G. (1982) *An Evolutionary Theory of Economic Change*, Harvard University Press: Cambridge, MA.

Nerlove, M. and Raut, L.K. (1997) 'Growth Models with Endogenous Population: A General Framework', *Handbook of Population and Family Economics*, vol. 1 edited by Rosenzweig, M.R. and Stark, O., North-Holland: Amsterdam.

Neugart, M. (2004) 'Complicated Dynamics in a Flow Model of the Labor Market', *Journal of Economic Behavior & Organization*, 53, 193–213.

Obstfeld, M. (1981) 'Capital Mobility and Devaluation in an Optimizing Model with Rational Expectations', *American Economic Review*, 71, 217–21.

Obstfeld, M. (1985) 'The Capital Inflows Problem Revisited: A Stylized Model of Southern Cone Disinflation', *Review of Economic Studies*, 52, 605–25.

Obstfeld, M. and Rogoff, K. (1984) 'Exchange Rate Dynamics with Sluggish Prices under Alternative Price-Adjustment Rules', *International Economic Review*, 25, 157–74.

Obstfeld, M. and Rogoff, K. (1995) 'Exchange Rate Dynamics Redux', *Journal of Political Economy*, 103, 624–60.

Obstfeld, M. and Rogoff, K. (1998) *Foundations of International Macroeconomics*, The MIT Press: Cambridge, MA.

Odedokum, M.O. (1999) 'How the Size of the Monetary Sector Affects Economic Growth: Econometric Evidence from Industrial and Developing Countries', *Journal of Policy Modeling*, 21, 213–41.

O'Hara, D.J. (1977) 'Location of Firms Within a Square Central Business District', *Journal of Political Economy*, 85, 1189–207.

Oniki, H. and Uzawa, H. (1965) 'Patterns of Trade and Investment in a Dynamic Model of International Trade', *Review of Economic Studies*, 32, 15–38.

Order, R. van (1977) 'Unemployment, Inflation, and Monetarism: A Further Analysis', *American Economic Review*, 67, 741–6.

Orphanides, A. (1997) 'Monetary Policy Rules Based on Real-Time Data', Finance and Economic Discussion Series No. 1998-03, *Federal Reserve Board*.

Orphanides, A. and Solow, R.M. (1990) 'Money, Inflation and Growth', *Handbook of Monetary Economics*, edited by Friedman, B.M. and Hahn, F.H. North-Holland: Amsterdam.

Palivos, T. (1995) 'Endogenous Fertility, Multiple Growth Paths, and Economic Convergence', *Journal of Economic Dynamics & Control*, 19, 1489–510.

Palivos, T. and Yip, C.V. (1995) 'Government Expenditure Financing in an Endogenous Growth Model: A Comparison', *Journal of Money, Credit and Banking*, 27, 1159–78.

Papageorgiou, Y.Y. and Smith, R.R. (1983) 'Agglomeration as Local Instability of Spatially Uniform Steady-States', *Econometrica*, 51, 1109–19.

Parente, S. (1994) 'Technology Adoption, Learning-by-Doing, and Economic Growth', *Journal of Economic Theory*, 63, 349–69.

Patinkin, D. (1956) *Money, Interest, and Prices*, Row, Peterson and Company: Evanston.

Patinkin, D. (1965) *Money, Interest, and Prices: An Interpretation of Money and Value Theory, 2nd edition*. Hamper & Row: New York.

Pecorino, P. (1995) 'Inflation, Human Capital Accumulation and Long-Run Growth', *Journal of Macroeconomics*, 17, 533–42.

Pemberton, J. (1985) 'A Model of Wage and Employment Dynamics with Endogenous Preferences', *Oxford Economic Papers*, 37, 448–65.

Pentecost, E. (2000) *Macroeconomics – An Open Economy Approach*, St. Martin's Press: New York.

Petrucci, A. (2003) 'Money, Endogenous Fertility and Economic Growth', *Journal of Macroeconomics*, 25, 527–39.

Phillips, A.W. (1958) 'The Relation Between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1861–1957', *Economica*, 25, 283–99.

Pigou, A.C. (1943) 'The Classical Stationary State', *The Economic Journal*, 53, 343–51.

Pissarides, C.A. (1990) *Equilibrium Unemployment Theory*, Basil Blackwell: Oxford.

Ploeg, F. van der and Alogoskoufis, G.S. (1994) 'Money and Endogenous Growth', *Journal of Money, Credit and Banking*, 26, 771–91.

Powers, D. (2005) 'Inside Money and the Effects of Inflation', *Journal of Macroeconomics*, 27, 494–516.

Rabenau, B.V. (1979) 'Urban Growth with Agglomeration Economics and Diseconomics', *Geographia Polonica* 42, 77–90.

Rader, T. (1963) 'The Existence of a Utility Function to Represent Preferences', *Review of Economic Studies*, 30, 229–32.

Rader, T. (1981) 'Utility over Time: The Homothetic Case', *Journal of Economic Theory*, 25, 219–36.

Ramsey, F. (1928) 'A Mathematical Theory of Saving', *Economic Journal*, 38, 543–59.

Rankin, N. and Roffia, B. (2003) 'Maximum Sustainable Government Debt in the Overlapping Generations Model', *Manchester School*, 71, 217–41.

Rauch, J.E. (1991) 'Reconciling the Pattern of Trade with the Pattern of Migration', *American Economic Review*, 81, 775–96.

Razin, A., Ben, A. and Zion, U. (1995) 'An Intergenerational Model of Population Growth', *American Economic Review*, 65, 923–33.

Rebelo, S. (1991) 'Long-Run Policy Analysis and Long-Run Growth', *Journal of Political Economy*, 99, 500–21.

Reis, R. (2007) 'The Analytics of Monetary Non-neutrality in the Sidrauski Model', *Economics Letters*, 94, 129–35.

Rey, S.J. and Janikas, M.V. (2005) 'Regional Convergence, Inequality, and Space', *Journal of Economic Geography*, 5, 155–76.

Richardson, B.V. (1973) *Regional Growth Theory*, John Wiley: New York.

Robert-Nicound, F. (2004) 'The Structure of Simple "New Economic Geography" Models (or, On Identical Twins)', *Journal of Economic Geography*, 5, 201–34.

Romer, D. (1996) *Advanced Macroeconomics*, The McGraw-Hill Companies: New York.

Romer, P.M. (1986) 'Increasing Returns and Long-Run Growth', *Journal of Political Economy*, 94, 1002–37.

Rosenzweig, M.R. and Stark, O. (eds) (1997) *Handbook of Population and Family Economics*, Elsevier: Amsterdam.

Rossi-Hansberg, E. and Wright, M.L.J. (2007) 'Urban Structure and Growth' *Review of Economic Studies*, 74, 597–624.

Roy, J.R. and Johansson, B. (1993) 'A Model of Trade Flows in Differentiated Goods', *The Annals of Regional Science*, 27, 95–115.

Rupert, P., Rogerson, R. and Wright, R. (2000) 'Homework in Labor Economics: Household Production and Intertemporal Substitution', *Journal of Monetary Economics*, 46, 557–79.

Ryder, H.E. and Heal, G.M. (1973) 'Optimal Growth with Intertemporally Dependent Preferences', *Review of Economic Studies*, 40, 1–31.

Samuelson, P.A. (1937) 'A Note on Measurement of Utility', *Review of Economic Studies*, 4, 155–61.

Samuelson, P.A. (1958) 'An Exact Consumption Loan Model of Interest with or without the Social Contrivance of Money', *Journal of Political Economy*, 66, 467–82.

Samuelson, P.A. (1965) 'A Theory of Induced Innovation Along Kennedy-Weizsäcker Lines', *Review of Economics and Statistics*, 47, 343–56.

Sargent, T.J. and Wallace, N. (1973) 'The Stability of Models of Money and Growth with Perfect Foresight', *Econometrica*, 41, 1043–8.

Sargent, T.J. and Wallace, N. (1975) 'Rational Expectations, the Optimal Monetary Instrument, and the Optimal Monetary Supply Rule', *Journal of Political Economy*, 83, 241–54.

Sargent, T.J. and Wallace, N. (1976) 'Rational Expectations and the Theory of Economic Policy', *Journal of Monetary Economics*, 2, 169–83.

Sato, R. and Tsutsui, S. (1984) 'Technical Progress, the Schumpeterian Hypothesis and Market Structure', *Journal of Economics*, S4, 1–37.

Saving, T.R. (1970) 'Outside Money, Inside Money, and the Real Balance Effect', *Journal of Money Credit and Banking*, 2, 83–100.

Saving, T.R. (1971) 'Transactions Costs and the Demand for Money', *American Economic Review*, 61, 407–402.

Saving, T.R. (1972) 'Transactions Costs and the Firms' Demand for Money', *Journal of Money, Credit and Banking*, 4, 245–59.

Saving, T.R. (1973) 'On the Neutrality of Money', *Journal of Political Economy*, 81, 98–119.

Schultz, T.W. (1981) *Investing in People – The Economics of Population Quality*, University of California Press: Berkeley.

Sener, F. (2001) 'Schumpeterian Unemployment, Trade and Wage', *Journal of International Economics*, 54m, 119–48.

Shell, K., Sidrauski, M. and Stiglitz, J.E. (1969) 'Capital Gains, Income and Saving', *Review of Economic Studies*, 36, 15–26.

Shi, S.Y. and Epstein, L.G. (1993) 'Habits and Time Preference', *International Economic Review*, 34, 61–84.

Shleifer, A. and Vishny, R.W. (1993) 'Corruption', *Quarterly Journal of Economics*, 108, 599–617.

Shone, R. (2002) 'Economic Dynamics – Phase Diagrams and Their Economic Application', Cambridge University Press: Cambridge.

Sidrauski M (1967a) 'Rational Choice and Patterns of Growth in a Monetary Economy', *American Economic Review*, 57, 534–44.

Sidrauski M (1967b) 'Inflation and Economic Growth', *Journal of Political Economy*, 75, 789–810.

Silos, P. (2006) 'Housing Portfolio Choice and the Macroeconomy', *Journal of Economic Dynamics & Control*, 31, 2774–801.

Simon, C.P. and Blume, L. (1994) *Mathematics for Economics*, W.W. Norton & Company: New York.

Simos, E.O. (1981) 'Real Money Balances as a Productive Input: Further Evidence', *Journal of Monetary Economics*, 7, 207–25.

Skinner, J. (1996) '*The Dynamic Efficiency Cost of Not Taxing Housing*', *Journal of Public Economics*, 59, 397–417.

Solow, R. (2000) *Growth Theory – An Exposition*, Oxford University Press: New York.
Solow, R.M. (1956) 'A Contribution to the Theory of Growth', *Quarterly Journal of Economics*, 70, 65–94.
Solow, R.M. (1972) 'Congestion, Density, and the Use of Land in Transportation', *Swedish Journal of Economics*, 74, 161–73.
Solow, R.M. (1973) 'On Equilibrium Models of Urban Location', *Essays in Modern Economics*, edited by Parkin, M., Longman: London.
Solow, R.M. (1979) 'Another Possible Source of Wage Stickiness'. *Journal of Macroeconomics*, 1, 79–82.
Solow, R.M. and Vickrey, W.S. (1971) 'Land Use in a Long Narrow City', *Journal of Economic Theory*, 3, 430–47.
Song, E.Y. (2002) 'Taxation, Human Capital and Growth', *Journal of Economic Dynamics* & Control, 26, 205–16.
Sonis, M., Hewings, G.J.D. and Okuyama, Y. (2001) 'Feedback Loops Analysis of Japanese Interregional Trade, 1980-85-90', *Journal of Economic Geography*, 1, 341–62.
Sorger, G. (2003) 'On the Multi-Country Version of the Solow-Swan Model', *Japanese Economic Review*, 54, 146–64.
Starr, R.M. (1974) 'The Price of Money in a Pure Exchange Monetary Economy with Taxation', *Ecnometrica*, 42, 45–54.
Stein, J.L. (1966) 'Money and Capacity Growth', *Journal of Political Economy*, 74, 451–65.
Stein, J.L. (1970) 'Monetary Growth Theory in Perspective', *American Economic Review*, 60, 85–106.
Stein, J.L. (1974) 'Unemployment, Inflation, and Monetarism', *American Economic Review*, 64, 867–87.
Stiglitz, J.E. (1967) 'A Two Sector Two Class Model of Economic Growth', *Review of Economic Studies*, 34, 227–38.
Stockman, A. (1981) 'Anticipated Inflation, and the Capital Stocks in a Cash-in-Advance Economy', *Journal of Monetary Economics*, 8, 387–93.
Stokey, N.L. (1988) 'Learning-by-Doing and the Introduction of New Goods', *Journal of Political Economy*, 96, 701–17.
Stokey, N.L. (1991) 'The Volume and Composition of Trade Between Rich and Poor Countries', *Review of Economic Studies*, 58, 63–80.
Suen, W., Chan, W. and Zhang, J. (2003) 'Martial Transfer and Intra-Household Allocation: A Nash-Bargaining Analysis', *Journal of Economic Behavior & Organization*, 52, 133–46.
Suen, M.H. and Yip, C.K. (2005) 'Superneutrality, Indeterminancy and Endogenous Growth', *Journal of Macroeconomics* 27, 579–95.
Sveen, T. and Weinke, L. (2005) 'New Perspectives on Capital, Sticky Prices, and the Taylor Principle', *Journal of Economic Theory*, 123, 21–39.
Svensson, L.E.O. (1997) 'Inflation Forecast Targeting: Implementing and Monitoring Inflation Targets', *European Economic Review*, 41, 1111–46.
Swan, T.W. (1956) 'Economic Growth and Capital Accumulation', *Economic* Record, 32, 334–61.
Tabuchi, T. and Thisse, J.F. (2002) 'Taste Heterogeneity, Labor Mobility and Economic Geography', *Journal of Development Economics*, 69, 155–77.
Takayama, A. (1985) *Mathematical Economics*, Cambridge University Press: Cambridge.
Takayama, T. and Judge, G.G. (1971) *Spatial and Temporal Price and Allocation Models*, North-Holland Publishing Company: Amsterdam.

Tataru, M. (1999) 'Growth Rates in Multidimensional Spatial Voting' *Mathematical Social Sciences*, 37, 253–63.

Taylor, J.B. (1977) 'Conditions for Unique Solutions in Stochastic Macroeconomic Models with Rational Expectations', *Econometrica*, 45, 1377–85.

Taylor, J.B. (1993) 'Discretion versus Policy Rules in Practice', *Cornegie-Rochester Conference Series on Public Policy*, 39, 195–214.

Tirole, J. (1985) 'Asset Bubbles and Overlapping Generations', *Econometrica*, 53, 1499–528.

Tobin, J. (1955) 'A Dynamic Aggregative Model', *Journal of Political Economy*, 63, 103–15.

Tobin, J. (1956) 'The Interest Elasticity of the Transactions Demand for Money', *Review of Economics and Statistics*, 38, 241–7.

Tobin, J. (1965) 'Money and Economic Growth', *Econometrica*, 33, 671–84.

Turnovsky, S.J. (1987) 'Monetary Growth, Inflation and Economic Activity in a Dynamic Macro Model', *International Economic Review*, 28, 207–30.

Turnovsky, S.J. (1997) *International Macroeconomic Dynamics*, The MIT Press: Cambridge, MA.

Turnovsky, S.J. (2000) *Methods of Macroeconomic Dynamics*, The MIT Press: Cambridge, MA.

Turnovsky, S. and Okuyama, T. (1994) 'Taxes, Housing, and Capital Accumulation in a Two-Sector Growing Economy', *Journal of Public Economics*, 53, 245–67.

Turnovsky, S.J. (1999) 'Fiscal Policy and Growth in a Small Open Economy with Elastic Labor Supply', *Canadian Journal of Economics*, 32, 1191–214.

Upton, C. (1981) 'An Equilibrium Model of City Size', *Journal of Urban Economics*, 10, 15–36.

Uzawa, H. (1965) 'Optimal Technical Change in an Aggregative Model of Economic Growth', *International Economic* Review, 6: 18–31.

Uzawa, H. (1968) 'Time Preference, the Consumption Function, and Optimum Asset Holdings', in *Capital and Growth*, edited by Wolfe, J.N., Aldine: Chicago.

Villanueva, D.P. (1971) 'A Neoclassical Monetary Growth Model with Independent Savings and Investment Functions', *Journal of Money, Credit and Banking*, 3, 750–9.

Von Thadden, L. (1999) *Money, Inflation, and Capital Formation*, Springer: Berlin.

Waller, C.J., Verdier, T. and Gardner, R. (2002) 'Corruption: Top Down or Bottom Up?', *Economic Inquiry*, 40, 688–703

Walsh, C.E. (2003) '*Monetary Theory and Policy*', The MIT Press: Cambridge, MA.

Wan, H.Y. (1970) 'Optimal Savings Programs under Intertemporally Dependent Preferences', *International Economic Review*, 11, 521– 47.

Wang, P. and Yip, C.K. (1992) 'Examining the Long-Run Effect on Economic Growth', *Journal of Macroeconomics*, 14, 359–69.

Wang, P., Yip, C.K. and Scotese, C. (1994) 'Fertility Choice and Economic Growth: Theory and Evidence', *Review of Economics and Statistics*, 76, 255–66.

Walsh, C.E. (2003) *Monetary Theory and Policy*, The MIT Press: Cambridge, MA.

Warner, J.T. and Pleeter, S. (2001) 'The Personal Discount Rate: Evidence from Military Downsizing Programs', *American Economic Review*, 91, 33–53.

Weil, P. (1987) 'Permanent Deficits and Inflation', *Journal of Monetary Economics*, 20, 394–410.

Weil, P. (1991) 'Is Money Net Wealth?', *International Economic Review*, 32, 37–53.

Weizsäcker, C.C. (1966) 'Tentative Notes on a Two-Sector Model with Induced Technical Progress', *Review of Economic Studies*, 33, 245–51.

Bank Performance

A Theoretical and Empirical Framework for the Analysis of Profitability, Competition and Efficiency

Jacob Bikker, De Nederlandsche Bank, the Netherlands and **Jaap W.B. Bos**, Utrecht School of Economics, the Netherlands

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Wetzell, D.L. (2002) 'On Some Unappreciated Implications of Becker's Time Allocation Model of Labor Supply', *Economics Letters*, 75, 219–25.

Wheaton, W.C. (2004) 'Commuting, Congestion, and Employment Dispersal in Cities with Mixed Land Use', *Journal of Urban Economics*, 55, 417–38.

Wilson, C. (1979) 'Anticipated Shocks and Exchange Rate Dynamics', *Journal of Political Economy*, 87, 639–47.

Yamamoto, K. (2005) 'A Two-Region Model with Two Types of Manufacturing Technologies and Agglomeration', *Regional Science and Urban Economics*, 35, 808–36.

Yip, C.K. and Zhang, J. (1997) 'A Simple Endogenous Growth Model with Endogenous Fertility: Indeterminacy and Uniqueness', *Journal of Population Economics*, 10, 97–110.

Yokoo, M. (2000) 'Chaotic Dynamics in a Two-Dimensional Overlapping Generations Model', *Journal of Economic Development and Control*, 24, 909–34.

You, J.S. (1981) 'Money, Technology, and the Production Function: An Empirical Study', *Canadian Journal of Economics*, 14, 515–24.

Zamagni, S. and Agliardi, E. (eds) (2004) *Time in Economic Theory*, Edward Elgar Publishing Co.: Vermont.

Zeeman, E.C. (1977) 'Catastrophe Theory: Selected Papers 1972–1977', Addison-Wesley: Reading, MA.

Zhang, W.B. (1990) '*Economic Dynamics – Growth and Development*', Springer-Verlag: Heidelberg.

Zhang, W.B. (1991) '*Synergetic Economics*', Springer-Verlag: Heidelberg.

Zhang, W.B. (1993a) 'Capital and Unemployment in a Two-Class Growth Model', *Seoul Journal of Economics*, 6, 361–77.

Zhang, W.B. (1993b) 'Location Choice and Land Use in an Isolated State – Endogenous Capital and Knowledge Accumulation', *Annals of Regional Science*, 27, 23–39.

Zhang, W.B. (1993c) 'Wages, Service Prices and Rent – Urban Division of Labor and Amenities', *Seoul Journal of Economics*, 6, 97–113.

Zhang, W.B. (1996) 'Economic Growth, Housing and Residential Location', Paper represented at The 14th Conference of the Pacific Regional Science Conference Organization, June 1995, Taipei.

Zhang, W.B. (1998a) *Japan versus China in the Industrial Race*, Macmillan: London.

Zhang, W.B. (1998b) 'Economic Geography with Two Regions – Capital Accumulation and Economic Structure', *Australian Economic Papers*, 35, 225–35.

Zhang, W.B. (1999) *Capital and Knowledge – Dynamics of Economic Structures with Non-constant Returns*, Springer-Verlag: Heidelberg.

Zhang, W.B. (2000) *A Theory of International Trade – Capital, Knowledge and Economic Structures*, Springer: Berlin.

Zhang, W.B. (2000a) 'Public Services, Increasing Returns and Equilibrium Dynamics', *Journal of Economic Dynamics and Control*, 24, 227–46.

Zhang, W.B. (2002a) *An Economic Theory of Cities – Spatial Models with Capital, Knowledge, and Structures*, Springer: Berlin.

Zhang, W.B. (2002b) *Singapore's Modernization – Westernization and Modernizing Confucian Manifestations*, Nova Science: New York.

Zhang, W.B. (2003a) *A Theory of Interregional Dynamics – Spatial Models with Capital, Knowledge, and Structures*, Springer: Berlin.

Zhang, W.B. (2003b) *Taiwan's Modernization*, World Scientific: Singapore.

Zhang, W.B. (2005a) *Economic Growth Theory*, Ashgate: Hampshire.

Zhang, W.B. (2005b) 'A Two-Sector Growth Model with Labour Supply', *Australian Journal of Labor Economics*, 8, 245–60.

Zhang, W.B. (2005c) *Differential Equations, Bifurcations, and Chaos in Economics*, World Scientific: Singapore.

Zhang, W.B. (2006a) *Growth with Income and Wealth Distribution*, London: Macmillan.

Zhang, W.B. (2006b) *Discrete Dynamical Systems, Bifurcations and Chaos in Economics*. Amsterdam: Elsevier.

Zhang, W.B (2006c) *Hong Kong - The Pearl Made of the British Mastery and the Chinese Docile-Diligence*, Nova Science: New York.

Zhang, W.B. (2007a) *New China's Long March from Servility to Freedom*, Nova Science: New York.

Zhang, W.B. (2007b) 'A Multi-Region Model with Capital Accumulation and Endogenous Amenities', *Environment and Planning*, A 39, 2248–70.

Zhang, W.B. (2007c) 'Growth and Agglomeration of a Small-Open Multi-Regional Economy', *Journal of Economic Integration*, 22, 515–49.

Zhang, W.B. (2008a) *International Trade Theory: Capital, Knowledge, Economic Structure, Money and Prices over Time and Space*, Springer: Berlin (forthcoming).

Zhang, W.B. (2008b) 'A Monetary Growth Model of a Small-Open Multi-Regional Economy', *Review of Development and Cooperation* (forthcoming).

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Zhang, W.B (2006c) *Hong Kong - The Pearl Made of the British Mastery and the Chinese Docile-Diligence*, Nova Science: New York.

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Zhang, W.B. (2008a) *International Trade Theory: Capital, Knowledge, Economic Structure, Money and Prices over Time and Space*, Springer: Berlin (forthcoming).

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Central Banking, Asset Prices and Financial Fragility

Éric Tymoigne, California State University, USA

The current literature on central banking contains two distinct branches. On the one side, research focuses on the impact of monetary policy on economic growth, unemployment, and output-price inflation, while ignoring financial aspects. On the other side, some scholars leave aside macroeconomics in order to study the narrow, but crucial, subjects of financial behaviours, and financial supervision and regulation.

Eric Tymoigne shows how different views regarding the conception of asset pricing lead to different positions regarding the appropriate role of a central bank in the economy. In addition, Hyman P. Minsky's framework of analysis is used extensively and is combined with other elements of the Post Keynesian framework to study the role of a central bank.

This book will be of interest to students and researchers engaged with central banking, macroeconomics, asset pricing and monetary economics.

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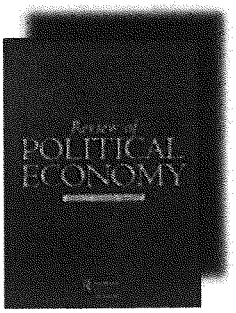

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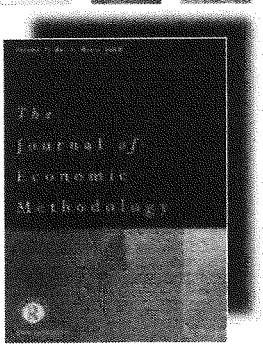

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